Chapter

2 Geometry

What you will learn

2A Geometry review (Consolidating)
2B Congruent triangles
2C Investigating parallelograms using congruence
2D Similar figures (Consolidating)
2E Proving similar triangles
2F Circles and chord properties (10A)
2G Angle properties of circles – theorems 1 and 2 (10A)
2H Angle properties of circles – theorems 3 and 4 (10A)
2I Tangents (Extending)
2J Intersecting chords, secants and tangents (Extending)

Australian curriculum

MEASUREMENT AND GEOMETRY

Geometric reasoning

Formulate proofs involving congruent triangles and angle properties.

Apply logical reasoning, including the use of congruence and similarity, to proofs and numerical exercises involving plane shapes.

(10A) Prove and apply angle and chord properties of circles.

Online resources

- Chapter pre-test
- Videos of all worked examples
- Interactive widgets
- Interactive walkthroughs
- Downloadable HOTsheets
- Access to HOTmaths Australian Curriculum courses

Burj Khalifa building, Dubai

Every building that people have walked into throughout the history of the civilised world was built with the help of geometry. The subject of geometry is intertwined with the structural engineering methods applied to the construction of all buildings. The ancient Greek mathematician and inventor Archimedes (born 287 BC) used simple geometry to design and build structures and machines to help advance society. Today, structural engineers design and build structures using complex mathematics and computer software based on the same geometrical principles. Very tall buildings like the Burj Khalifa in Dubai, which is 828 m tall, need to be designed to take into account all sorts of engineering problems – earthquakes, wind, sustainability and aesthetics, to name just a few.

2A Geometry review CONSOLIDATING

Interactive

Widgets

HOTsheets

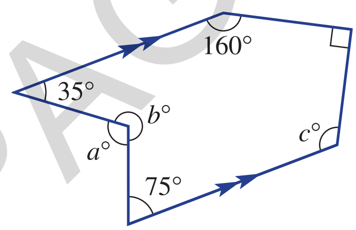
Walkthroughs

Based on just five simple axioms (i.e. known or self-evident facts) the famous Greek mathematician Euclid (about 300 BC) was able to deduce many hundreds of propositions (theorems and constructions) systematically presented in the 13-volume book collection called the *Elements*. All the basic components of school geometry are outlined in these books, including the topics *angle sums of polygons* and *angles in parallel lines*, which are studied in this section.

Let's start: Three unknown angles

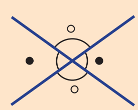
This hexagon contains a pair of parallel sides and three unknown angles, a, b and c.

- Find the value of a using the given angles in the hexagon. *Hint:* Add a construction line parallel to the two parallel sides so that the angle of size $a°$ is divided into two smaller angles. Give reasons throughout your solution.
- Find the value of b, giving a reason.
- What is the angle sum of a hexagon? Use this to find the value of c.
- Can you find a different method to find the value of c, using parallel lines?

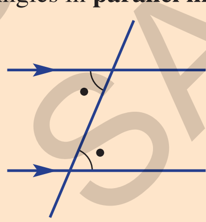


Key ideas

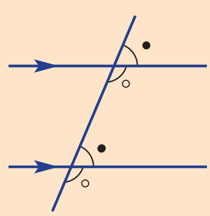
- Angles at a point
  - **Complementary** (sum to 90°)
  - **Supplementary** (sum to 180°)
  - **Revolution** (360°)
  - **Vertically opposite angles** (equal)



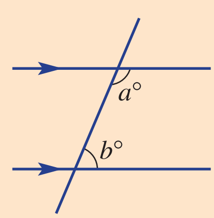
- Angles in **parallel lines**



Alternate angles are equal.



Corresponding angles are equal.

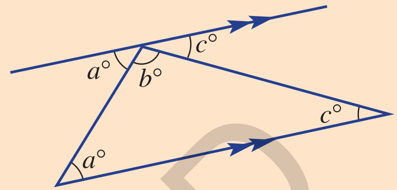


Cointerior angles are supplementary.
$a + b = 180$

  - If two lines, AB and CD, are parallel, we write $AB \parallel CD$.

Key ideas

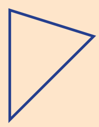
- **Triangles**
  - Angle sum is 180°.
    To prove this, draw a line parallel to a base and then mark the alternate angles in parallel lines. Note that angles on a straight line are supplementary.



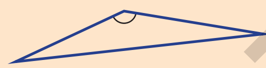
$a + b + c = 180$

  - Triangles classified by angles.

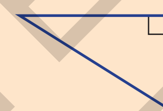
Acute: all angles acute



Obtuse: one angle obtuse

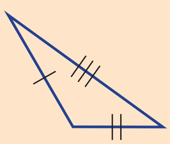


Right: one right angle

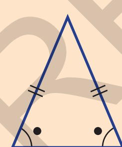


  - Triangles classified by side lengths.

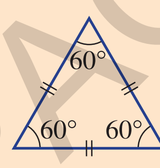
Scalene (3 different sides)



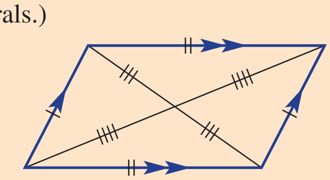
Isosceles (2 equal sides)



Equilateral (3 equal sides)



- **Quadrilaterals** (Refer to section 2C for more details on quadrilaterals.)
  - **Parallelograms** are quadrilaterals with two pairs of parallel sides.
  - **Rectangles** are parallelograms with all angles 90°.
  - **Rhombuses** are parallelograms with sides of equal length.
  - **Squares** are parallelograms that are both rectangles and rhombuses.
  - **Kites** are quadrilaterals with two pairs of equal adjacent sides.
  - **Trapeziums** are quadrilaterals with at least one pair of parallel sides.



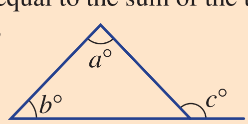
- **Polygons** have an angle sum given by $S = 180(n - 2)$, where n is the number of sides.
  - **Regular polygons** have equal sides and equal angles.

    A single interior angle $= \dfrac{180(n-2)}{n}$

n	Name
3	triangle
4	quadrilateral
5	pentagon
6	hexagon
7	heptagon
8	octagon
9	nonagon
10	decagon

- An **exterior angle** is supplementary to an interior angle.
  - For a triangle, the **exterior angle theorem** states that the exterior angle is equal to the sum of the two opposite interior angles.

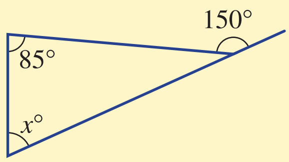
    $c = a + b$



Example 1 Using exterior angles and angle sums

Find the value of x in the following, giving reasons.

a



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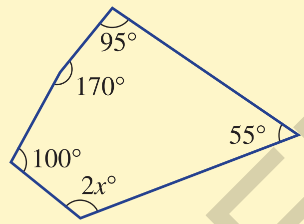
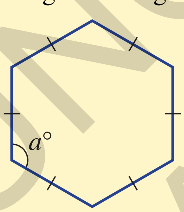


SOLUTION	EXPLANATION
a $x + 85 = 150$ (exterior angle theorem) $x = 65$	Use the exterior angle theorem for a triangle.
b $S = 180(n - 2)$ $= 180 \times (5 - 2)$ $= 540$	Use the rule for the angle sum of a polygon (5 sides, so $n = 5$).
$2x + 100 + 170 + 95 + 55 = 540$ $2x + 420 = 540$ $2x = 120$ $\therefore\ x = 60$	The sum of all the angles is 540°.

Example 2 Working with regular polygons and parallel lines

Find the value of the pronumeral, giving reasons.

a a regular hexagon



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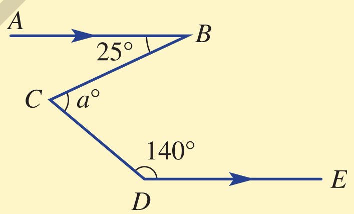


SOLUTION	EXPLANATION
a $S = 180(n - 2)$ $= 180 \times (6 - 2)$ $= 720$	Use the angle sum rule for a polygon with $n = 6$.
$a = 720 \div 6$ $= 120$	In a regular hexagon there are 6 equal angles.

b Construct a third parallel line, CF.

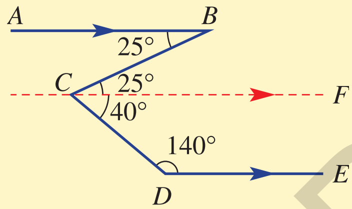
$\angle BCF = 25^\circ$ (alternate angles in ∥ lines)

$\angle FCD = 180^\circ - 140^\circ$

$= 40^\circ$ (cointerior angles in ∥ lines)

$\therefore\ a = 25 + 40$

$= 65$



$\angle ABC$ and $\angle FCB$ are alternate angles in parallel lines. $\angle FCD$ and $\angle EDC$ are cointerior angles in parallel lines.

Exercise 2A

1–3	3	—

UNDERSTANDING

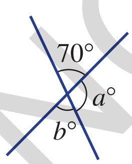
1 List the names of the polygons with 3 to 10 sides, inclusive.

2 Decide whether each of the following is true or false.

- **a** The angle sum of a quadrilateral is 300°.
- **b** A square has 4 lines of symmetry.
- **c** An isosceles triangle has two equal sides.
- **d** An exterior angle on an equilateral triangle is 120°.
- **e** A kite has two pairs of equal opposite angles.
- **f** A parallelogram is a rhombus.
- **g** A square is a rectangle.
- **h** Vertically opposite angles are supplementary.
- **i** Cointerior angles in parallel lines are supplementary.

3 Find the values of the pronumerals, giving reasons.

a



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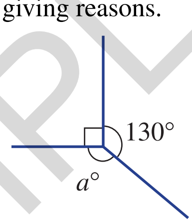

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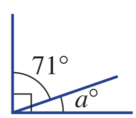

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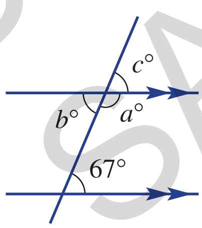

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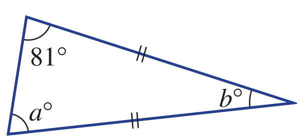

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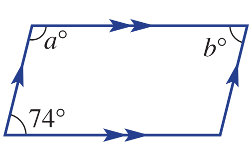
g



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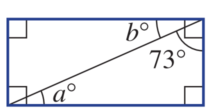
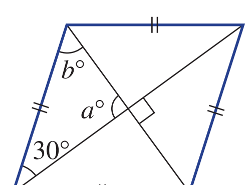


i



2A

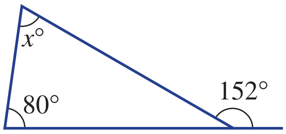
4, 5–6(½), 7	4, 5–6(½), 7	4, 5–6(½), 7

FLUENCY

Example 1a

4 Using the exterior angle theorem, find the value of the pronumeral.

a



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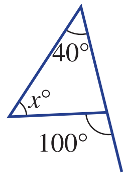

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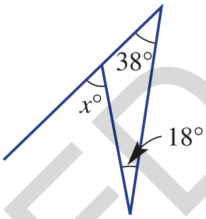
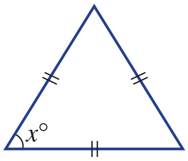


5 Find the value of the pronumeral, giving reasons.

a



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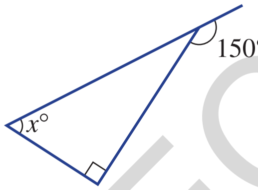

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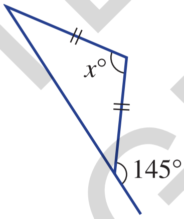

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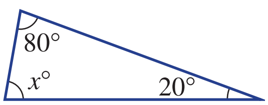

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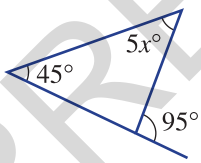

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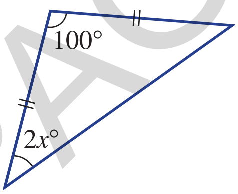

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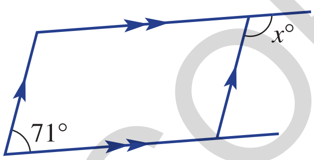

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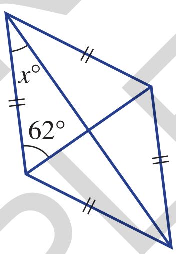

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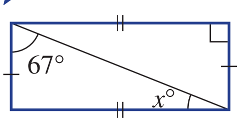

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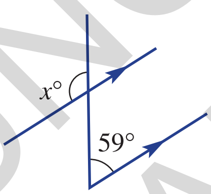

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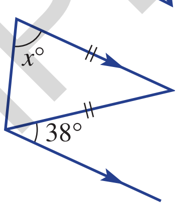

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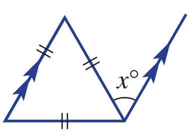

Example 1b

6 Find the value of x in the following, giving reasons.

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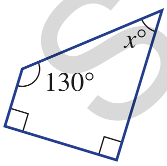

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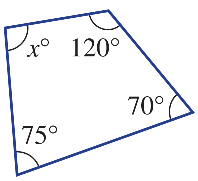

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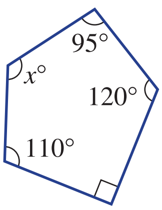

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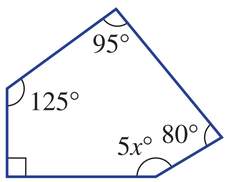
d



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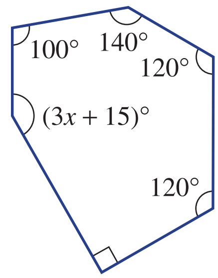

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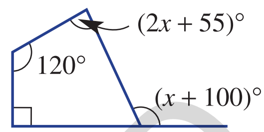


Example 2a

7 Find the size of an interior angle of these polygons if they are regular.

a pentagon **b** octagon **c** decagon

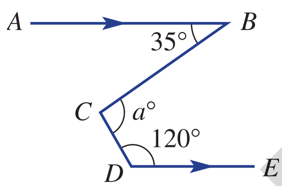
8(½), 9, 10 | 8(½), 9–11 | 8(½), 10–12

PROBLEM-SOLVING

Example 2b

8 Find the value of the pronumeral a, giving reasons.

a



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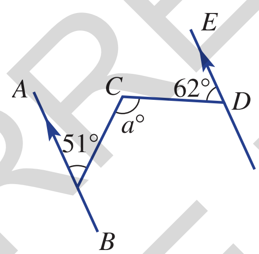

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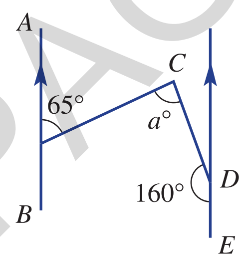

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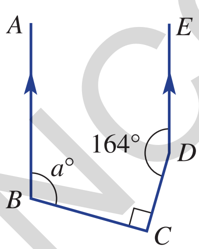

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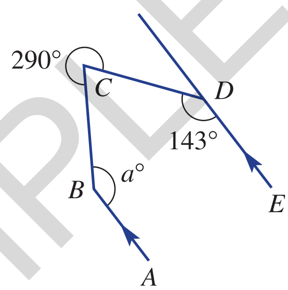

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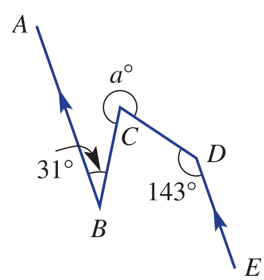


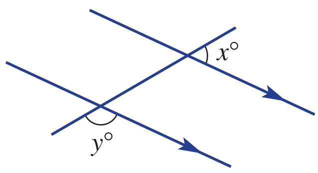
9 **a** Find the size of an interior angle of a regular polygon with 100 sides.

b What is the size of an exterior angle of a 100-sided regular polygon?

10 Find the number of sides of a polygon that has the following interior angles.

a 150° **b** 162° **c** 172.5°

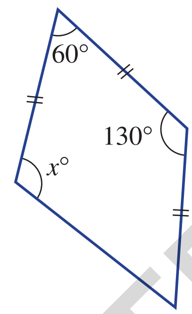
11 In this diagram, $y = 4x$. Find the values of x and y.



2A

PROBLEM-SOLVING

12 Find the value of x in this diagram, giving reasons. (*Hint:* Form isosceles and/or equilateral triangles.)



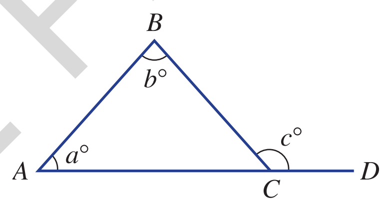
REASONING

13, 14 | 13–15 | 14–17

13 The rule for the sum of the interior angles of a polygon, $S°$, is given by $S = 180(n - 2)$.

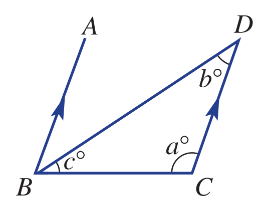
- **a** Show that $S = 180n - 360$.
- **b** Find a rule for the number of sides n of a polygon with an angle sum S; i.e. write n in terms of S.
- **c** Write the rule for the size of an interior angle $I°$ of a regular polygon with n sides.
- **d** Write the rule for the size of an exterior angle $E°$ of a regular polygon with n sides.

14 Prove that the exterior angle of a triangle is equal to the sum of the two opposite interior angles by following these steps.

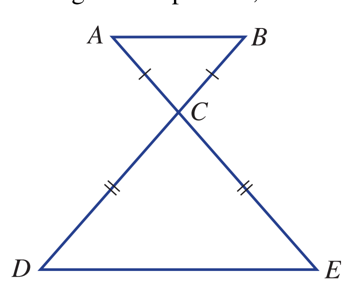


- **a** Write $\angle BCA$ in terms of a and b and give a reason.
- **b** Find c in terms of a and b using $\angle BCA$ and give a reason.

15 **a** Explain why in this diagram $\angle ABD$ is equal to $b°$.
- **b** Using $\angle ABC$ and $\angle BCD$, what can be said about a, b and c?
- **c** What does your answer to part **b** show?



16 Give reasons why AB and DE in this diagram are parallel; i.e. $AB \parallel DE$.



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2A

REASONING

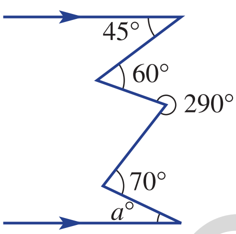
17 Each point on the Earth's surface can be described by a line of longitude (degrees east or west from Greenwich, England) and a line of latitude (degrees north or south from the equator). Investigate and write a brief report (providing examples) describing how places on the Earth can be located with the use of longitude and latitude.

Multilayered reasoning — — 18–20

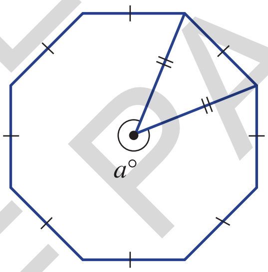
ENRICHMENT

18 Find the value of the pronumerals, giving reasons.

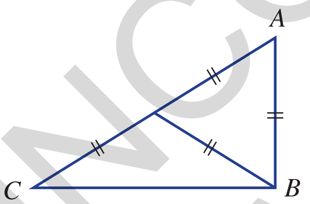
a



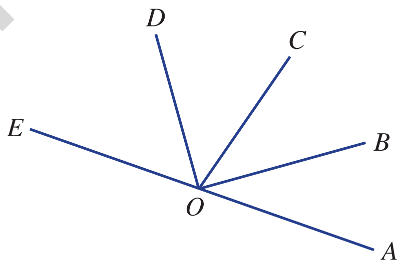
b



19 Give reasons why $\angle ABC = 90°$.



20 In this diagram $\angle AOB = \angle BOC$ and $\angle COD = \angle DOE$. Give reasons why $\angle BOD = 90°$.



Exploring triangles with dynamic geometry

1 Construct a triangle and illustrate the angle sum and the exterior angle theorem.

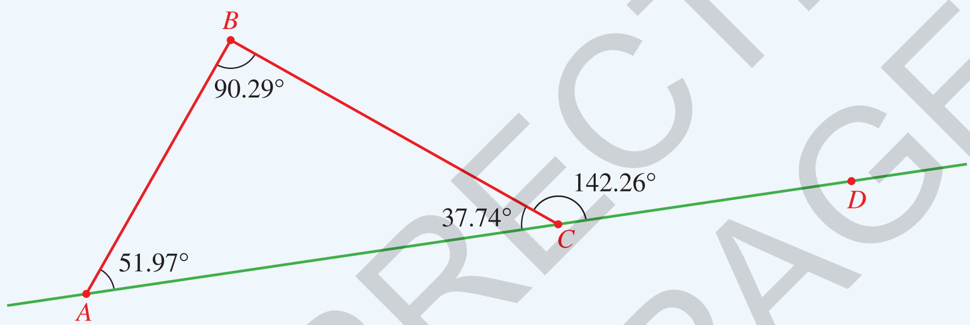
Dynamic geometry instructions and screens

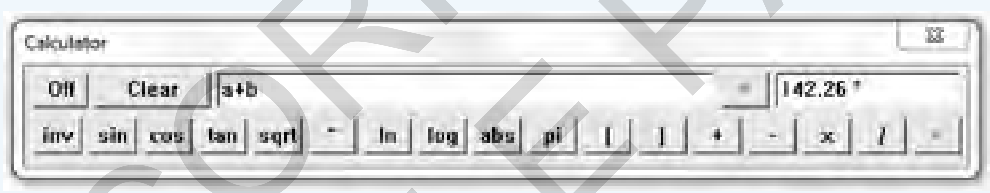
a Construct a line AD and triangle ABC, as shown.

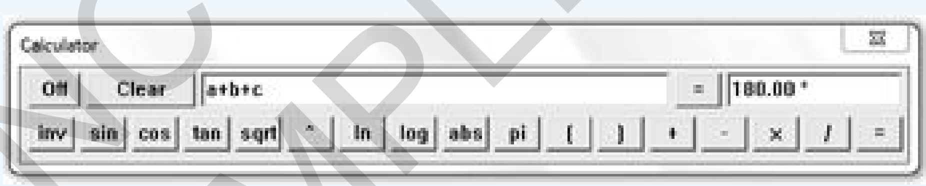
b Measure all the interior angles and the exterior angle $\angle BCD$.

c Use the calculator to check that the angle sum is 180° and that $\angle BCD = \angle BAC + \angle ABC$.

d Drag one of the points to check that these properties are retained for all triangles.







2B Congruent triangles

Interactive
Widgets
HOTsheets
Walkthroughs

In geometry it is important to know whether or not two objects are in fact identical in shape and size. If two objects are identical, then we say they are congruent. Two shapes that are congruent will have corresponding (i.e. matching) sides equal in length and corresponding angles also equal. For two triangles it is not necessary to know every side and angle to determine if they are congruent. Instead, a set of minimum conditions is enough. There are four sets of minimum conditions for triangles and these are known as the tests for congruence of triangles.

Are the Deira Twin Towers in Dubai congruent?

Let's start: Which are congruent?

Consider these four triangles.

1 $\triangle ABC$ with $\angle A = 37°$, $\angle B = 112°$ and $AC = 5$ cm.
2 $\triangle DEF$ with $\angle D = 37°$, $DF = 5$ cm and $\angle E = 112°$.
3 $\triangle GHI$ with $\angle G = 45°$, $GH = 7$ cm and $HI = 5$ cm.
4 $\triangle JKL$ with $\angle J = 45°$, $JK = 7$ cm and $KL = 5$ cm.

Sarah says that only $\triangle ABC$ and $\triangle DEF$ are congruent. George says that only $\triangle GHI$ and $\triangle JKL$ are congruent and Tobias says that both pairs ($\triangle ABC$, $\triangle DEF$ and $\triangle GHI$, $\triangle JKL$) are congruent.

- Discuss which pairs of triangles might be congruent, giving reasons.
- What drawings can be made to support your argument?
- Who is correct: Sarah, George or Tobias? Explain why.

Key ideas

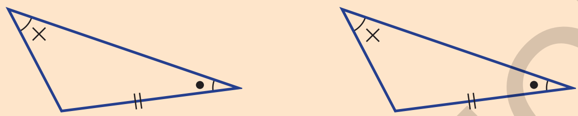
- Two objects are said to be **congruent** when they are exactly the same size and shape. For two congruent triangles $\triangle ABC$ and $\triangle DEF$, we write $\triangle ABC \equiv \triangle DEF$.
  - When comparing two triangles, corresponding sides are equal in length and corresponding angles are equal.
  - When we prove congruence in triangles, we usually write vertices in matching order.

Key ideas

- Two triangles can be tested for **congruence** using the following conditions.
  - Corresponding sides are equal (SSS).

  - Two corresponding sides and the included angle are equal (SAS).

  - Two corresponding angles and a side are equal (AAS).



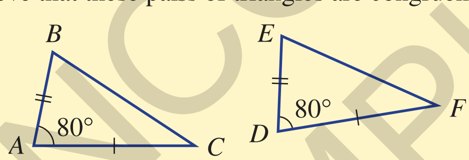
  - A right angle, the hypotenuse and one other pair of corresponding sides are equal (RHS).

- $AB \parallel CD$ means AB is parallel to CD.
- $AB \perp CD$ means AB is perpendicular to CD.

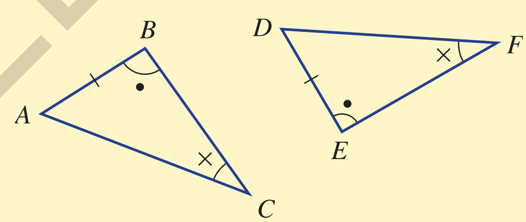
Example 3 Proving congruence in triangles

Prove that these pairs of triangles are congruent.

a



b



SOLUTION

a $AB = DE$ (given) S

$\angle BAC = \angle EDF = 80^\circ$ (given) A

$AC = DF$ (given) S

So, $\triangle ABC \equiv \triangle DEF$ (SAS)

EXPLANATION

List all pairs of corresponding sides and angles.

The two triangles are therefore congruent, with two pairs of corresponding sides and the included angle equal.

SOLUTION

b $\angle ABC = \angle DEF$ (given) A

$\angle ACB = \angle DFE$ (given) A

$AB = DE$ (given) S

So, $\triangle ABC \equiv \triangle DEF$ (AAS)

EXPLANATION

List all pairs of corresponding sides and angles.

The two triangles are therefore congruent with two pairs of corresponding angles and a corresponding side equal.

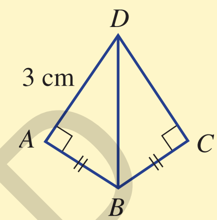
Example 4 Using congruence in proof

In this diagram, $\angle A = \angle C = 90^\circ$ and $AB = CB$.

a Prove $\triangle ABD \equiv \triangle CBD$.

b Prove $AD = CD$.

c State the length of CD.



SOLUTION	EXPLANATION
a $\angle A = \angle C = 90^\circ$ (given) R BD is common. H $AB = CB$ (given) S $\therefore \triangle ABD \equiv \triangle CBD$ (RHS)	Systematically list corresponding pairs of equal angles and lengths.
b $\triangle ABD \equiv \triangle CBD$ so $AD = CD$ (corresponding sides in congruent triangles)	Since $\triangle ABD$ and $\triangle CBD$ are congruent, the matching sides AD and CD are equal.
c $CD = 3$ cm	$AD = CD$ from part **b** above.

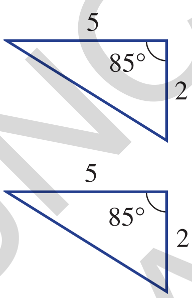
Exercise 2B

1, 2	1, 2	—

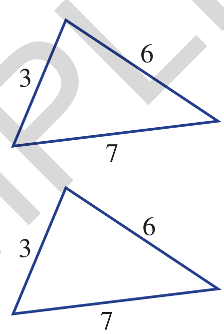
UNDERSTANDING

1 Which of the tests (SSS, SAS, AAS or RHS) would be used to decide whether the following pairs of triangles are congruent?

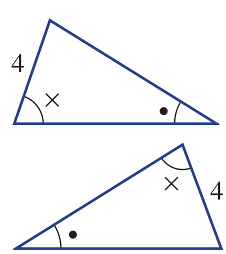
a



b



c



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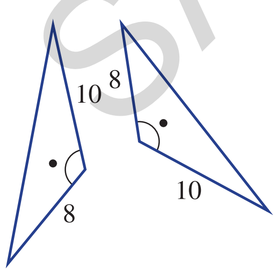

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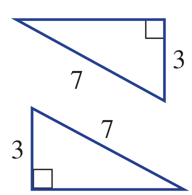

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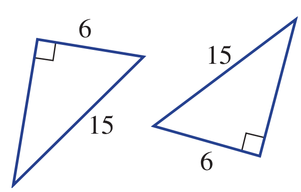
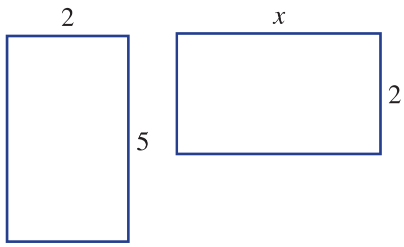


2B

UNDERSTANDING

2 Assume these pairs of figures are congruent and find the value of the pronumeral in each case.

a



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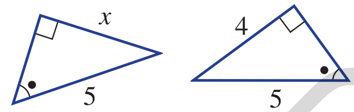

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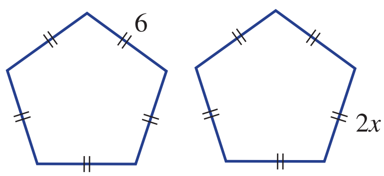

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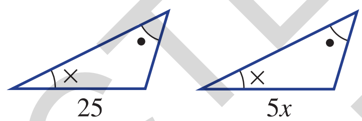

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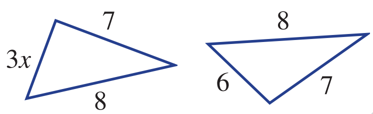

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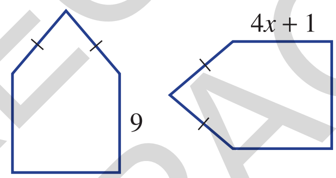


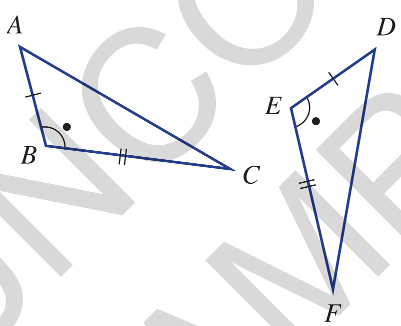
3, 4 | 3, 4–5(½) | 3–5(½)

FLUENCY

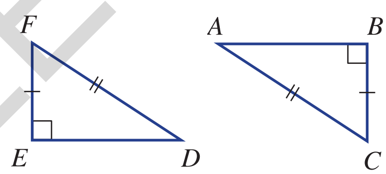
Example 3

3 Prove that these pairs of triangles are congruent.

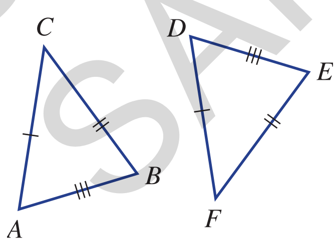
a



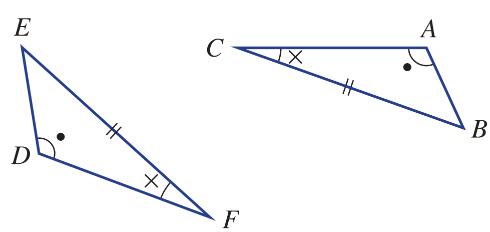
b



c



d

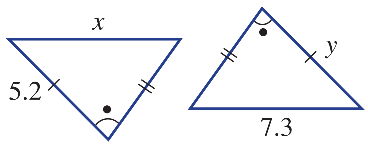


2B

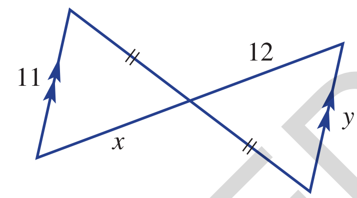
FLUENCY

4 Find the value of the pronumerals in these diagrams, which include congruent triangles.

a



b



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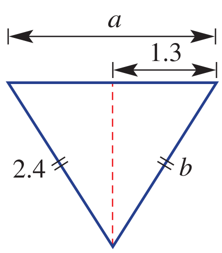

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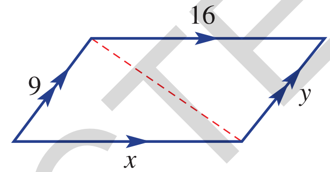
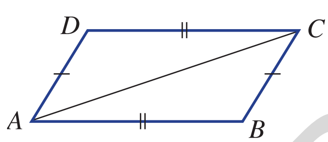


5 Prove that each pair of triangles is congruent, giving reasons. Write the vertices in matching order.

a



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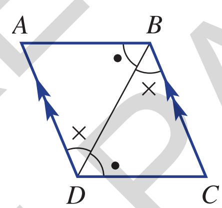

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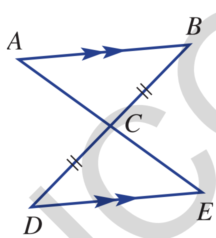

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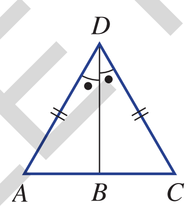

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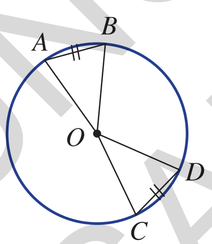

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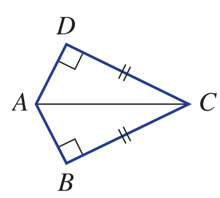


6	6, 7	6–8

PROBLEM-SOLVING

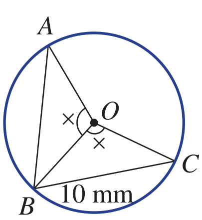
Example 4

6 In this diagram, O is the centre of the circle and $\angle AOB = \angle COB$.

a Prove $\triangle AOB \equiv \triangle COB$.

b Prove $AB = BC$.

c State the length of AB.



2B

PROBLEM-SOLVING

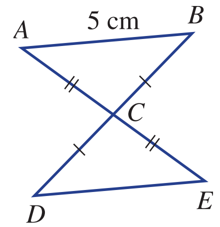
7 In this diagram, $BC = DC$ and $AC = EC$.

a Prove $\triangle ABC \equiv \triangle EDC$.

b Prove $AB = DE$.

c Prove $AB \parallel DE$.

d State the length of DE.

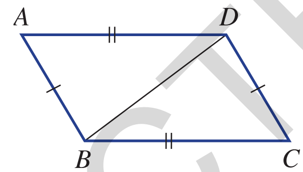


8 In this diagram, $AB = CD$ and $AD = CB$.

a Prove $\triangle ABD \equiv \triangle CDB$.

b Prove $\angle DBC = \angle BDA$.

c Prove $AD \parallel BC$.

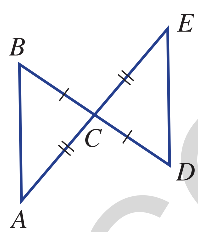


REASONING

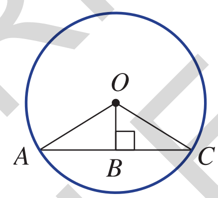
9a,b	9(½)	9

9 Prove the following for the given diagrams. Give reasons.

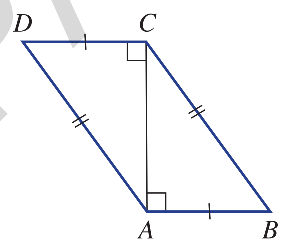
a $AB \parallel DE$



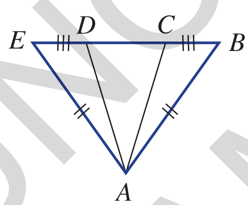
b OB bisects AC; i.e. $AB = BC$



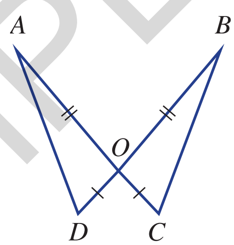
c $AD \parallel BC$



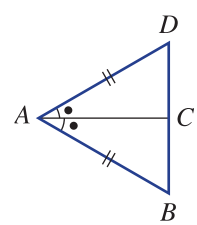
d $AD = AC$



e $\angle OAD = \angle OBC$



f $AC \perp BD$; i.e. $\angle ACD = \angle ACB$



ENRICHMENT

Draw your own diagram

—	—	10

10 a A circle with centre O has a chord AB. M is the midpoint of the chord AB. Prove $OM \perp AB$.

b Two overlapping circles with centres O and C intersect at A and B. Prove $\angle AOC = \angle BOC$.

c $\triangle ABC$ is isosceles with $AC = BC$, D is a point on AC such that $\angle ABD = \angle CBD$, and E is a point on BC such that $\angle BAE = \angle CAE$. AE and BD intersect at F. Prove $AF = BF$.

2C Investigating parallelograms using congruence

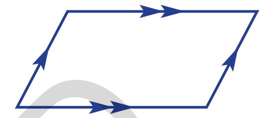
Interactive

Widgets

HOTsheets

Walkthroughs

Recall that parallelograms are quadrilaterals with two pairs of parallel sides. We therefore classify rectangles, squares and rhombuses as special types of parallelograms, the properties of which can be explored using congruence. By dividing a parallelogram into 'smaller' triangles, we can prove congruence for pairs of these triangles and use this to deduce the properties of the shape.



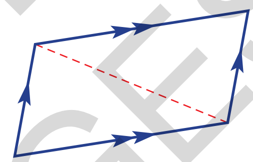
Let's start: Aren't they the same proof?

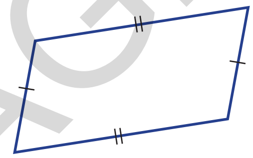
Here are two statements involving the properties of a parallelogram.

1 A parallelogram (with parallel opposite sides) has opposite sides of equal length.

2 A quadrilateral with opposite sides of equal length is a parallelogram.

- Are the two statements saying the same thing?
- Discuss how congruence can be used to help prove each statement.
- Formulate a proof for each statement.





Key ideas

- Some vocabulary and symbols:
  - If AB is parallel to BC, then we write $AB \parallel CD$.
  - If AB is perpendicular to CD, then we write $AB \perp CD$.
  - To bisect means to cut in half.
- **Parallelogram** properties and tests:
  - Parallelogram – a quadrilateral with opposite sides parallel.

Properties of a parallelogram	Tests for a parallelogram
- opposite sides are equal in length - opposite angles are equal - diagonals bisect each other	- if opposite sides are equal in length - if opposite angles are equal - if one pair of opposite sides are equal and parallel - if diagonals bisect each other

  - **Rhombus** – a parallelogram with all sides of equal length.

Properties of a rhombus	Tests for a rhombus
- all sides are equal length - opposite angles are equal - diagonals bisect each other at right angles - diagonals bisect the interior angles	- if all sides are equal length - if diagonals bisect each other at right angles

  - **Rectangle** – a parallelogram with all angles 90°.

Properties of a rectangle	Tests for a rectangle
- opposite sides are of equal length - all angles equal 90° - diagonals are equal in length and bisect each other	- if all angles are 90° - if diagonals are equal in length and bisect each other

Key ideas

- **Square** – a parallelogram that is a rectangle and a rhombus.

Properties of a square	Tests for a square
- all sides are equal in length - all angles equal 90° - diagonals are equal in length and bisect each other at right angles - diagonals bisect the interior angles	- if all sides are equal in length and at least one angle is 90° - if diagonals are equal in length and bisect each other at right angles

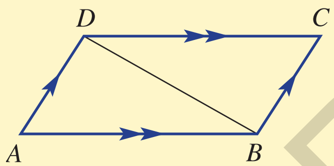
Example 5 Proving properties of quadrilaterals

a Prove that a parallelogram (with opposite sides parallel) has equal opposite angles.

b Use the property that opposite sides of a parallelogram are equal to prove that a rectangle (with all angles 90°) has diagonals of equal length.

SOLUTION / EXPLANATION

a



Draw a parallelogram with parallel sides and the segment *BD*.

$\angle ABD = \angle CDB$ (alternate angles in ∥ lines) A
$\angle ADB = \angle CBD$ (alternate angles in ∥ lines) A

Prove congruence of $\triangle ABD$ and $\triangle CDB$, noting alternate angles in parallel lines.

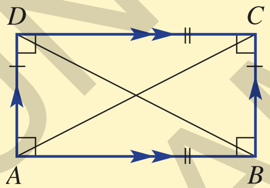
BD (common) S
$\therefore \triangle ABD \equiv \triangle CDB$ (AAS)

Note also that *BD* is common to both triangles.

$\therefore \angle DAB = \angle BCD$, so opposite angles are equal.

Corresponding angles in congruent triangles.

b



First, draw a rectangle with the given properties.

Consider $\triangle ABC$ and $\triangle BAD$.
AB (common) S
$\angle ABC = \angle BAD = 90°$ A

Choose $\triangle ABC$ and $\triangle BAD$, which each contain one diagonal.

$BC = AD$ (opposite sides of a parallelogram are equal in length) S
$\therefore \triangle ABC \equiv \triangle BAD$ (SAS)

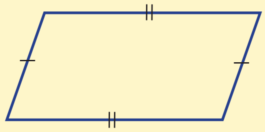
Prove congruent triangles using SAS.

$\therefore AC = BD$, so diagonals are of equal length.

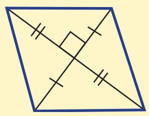
Corresponding sides in congruent triangles.

Example 6 Testing for a type of quadrilateral

a Prove that if opposite sides of a quadrilateral are equal in length, then it is a parallelogram.

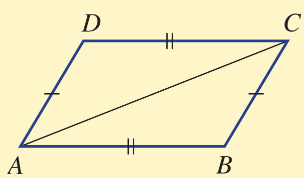


b Prove that if the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.



SOLUTION | **EXPLANATION**

a



First, label your quadrilateral and choose two triangles, $\triangle ABC$ and $\triangle CDA$.

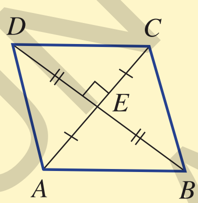
$AB = CD$ (given) S
$BC = DA$ (given) S
AC (common) S
$\therefore\ \triangle ABC \equiv \triangle CDA$ (SSS)

Prove that they are congruent using SSS.

$\therefore\ \angle BAC = \angle DCA$
So $AB \parallel DC$ (since alternate angles are equal).
Also $\angle ACB = \angle CAD$.
$\therefore\ AD \parallel BC$ (since alternate angles are equal).
$\therefore\ ABCD$ is a parallelogram.

Choose corresponding angles in the congruent triangles to show that opposite sides are parallel. If alternate angles between lines are equal then the lines must be parallel.

b



All angles at the point E are 90°, so it is easy to prove congruency of all four smaller triangles using SAS.

$\triangle ABE \equiv \triangle CBE \equiv \triangle ADE \equiv \triangle CDE$ by SAS
$\therefore\ AB = CB = CD = DA$

Corresponding sides in congruent triangles.
All sides are therefore equal.

and since $\angle ABE = \angle CDE$,
then $AB \parallel DC$.
Also, as $\angle CBE = \angle ADE$,
then $AD \parallel BC$.

Prove opposite sides are parallel using pairs of equal alternate angles.

$\therefore\ ABCD$ is a rhombus.

All sides are equal and opposite sides parallel.

Exercise 2C

1–3	2, 3	—

UNDERSTANDING

1 Name the special quadrilateral given by these descriptions.

- **a** a parallelogram with all angles 90°
- **b** a quadrilateral with opposite sides parallel
- **c** a parallelogram that is a rhombus and a rectangle
- **d** a parallelogram with all sides of equal length

2 List all the special quadrilaterals that have these properties.

- **a** All angles 90°.
- **b** Diagonals are equal in length.
- **c** Diagonals bisect each other.
- **d** Diagonals bisect each other at 90°.
- **e** Diagonals bisect the interior angles.

3 Give a reason why:

- **a** A trapezium is not a parallelogram.
- **b** A kite is not a parallelogram.

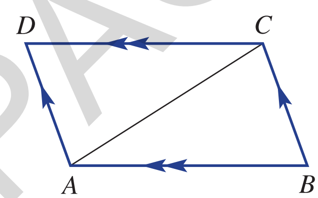
4, 5	4–6	4–6

FLUENCY

Example 5

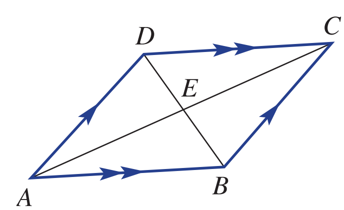
4 Complete these steps to prove that a parallelogram (with opposite parallel sides) has equal opposite sides.

- **a** Prove $\triangle ABC \equiv \triangle CDA$.
- **b** Hence, prove opposite sides are equal.



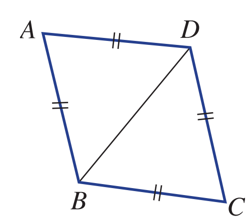
5 Complete these steps to prove that a parallelogram (with opposite equal parallel sides) has diagonals that bisect each other.

- **a** Prove $\triangle ABE \equiv \triangle CDE$.
- **b** Hence, prove $AE = CE$ and $BE = DE$.



6 Complete these steps to prove that a rhombus (with sides of equal length) has diagonals that bisect the interior angles.

- **a** Prove $\triangle ABD \equiv \triangle CDB$.
- **b** Hence, prove BD bisects both $\angle ABC$ and $\angle CDA$.



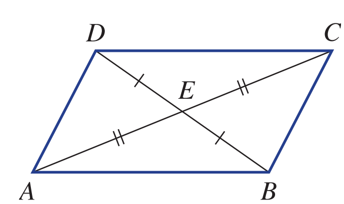
7	7, 8	8, 9

PROBLEM-SOLVING

Example 6

7 Complete these steps to prove that if the diagonals in a quadrilateral bisect each other, then it is a parallelogram.

- **a** Prove $\triangle ABE \equiv \triangle CDE$.
- **b** Hence, prove $AB \parallel DC$ and $AD \parallel BC$.

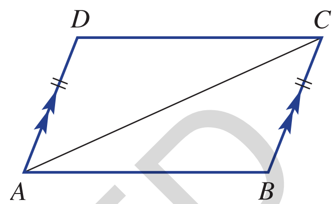


PROBLEM-SOLVING

8 Complete these steps to prove that if one pair of opposite sides is equal and parallel in a quadrilateral, then it is a parallelogram.

a Prove $\triangle ABC \equiv \triangle CDA$.

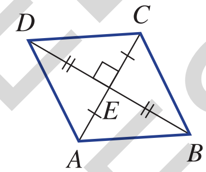
b Hence, prove $AB \parallel DC$.



9 Complete these steps to prove that if the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.

a Give a brief reason why $\triangle ABE \equiv \triangle CBE \equiv \triangle ADE \equiv \triangle CDE$.

b Hence, prove $ABCD$ is a rhombus.



10	10, 11	11, 12

REASONING

10 Prove that the diagonals of a rhombus (i.e. a parallelogram with sides of equal length):

a intersect at right angles

b bisect the interior angles.

11 Prove that a parallelogram with one right angle is a rectangle.

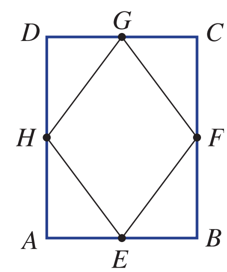
12 Prove that if the diagonals of a quadrilateral bisect each other and are of equal length, then it is a rectangle.

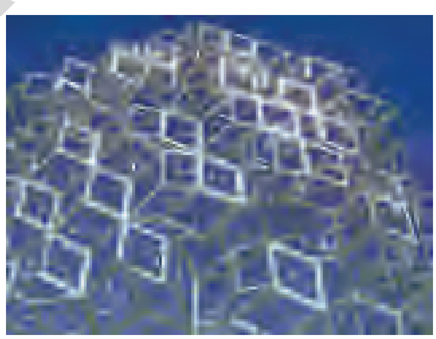
Rhombus in a rectangle

—	—	13

ENRICHMENT

13 In this diagram, E, F, G and H are the midpoints of AB, BC, CD and DA, respectively, and $ABCD$ is a rectangle.
Prove that $EFGH$ is a rhombus.





2D Similar figures CONSOLIDATING

Interactive
Widgets
HOTsheets
Walkthroughs

You will recall that the transformation called enlargement involves reducing or enlarging the size of an object. The final image will be of the same shape but of different size. This means that matching pairs of angles will be equal and matching sides will be in the same ratio, just as in an accurate scale model.

Let's start: The Q1 tower

The Q1 tower, pictured here, is located on the Gold Coast and was the world's tallest residential tower up until 2011. It is 245 m tall.

- Measure the height and width of the Q1 tower in this photograph.
- Can a scale factor for the photograph and the actual Q1 tower be calculated? How?
- How can you calculate the actual width of the Q1 tower using this photograph? Discuss.

Key ideas

- **Similar figures** have the same shape but are of different size.
  - Corresponding angles are equal.
  - Corresponding sides are in the same proportion (or ratio).
- **Scale factor** $= \dfrac{\text{image length}}{\text{original length}}$
- The symbols ||| or ~ are used to describe similarity and to write similarity statements. For example, $\triangle ABC \; ||| \; \triangle DEF$ or $\triangle ABC \sim \triangle DEF$.

Example 7 Finding and using scale factors

These two shapes are similar.

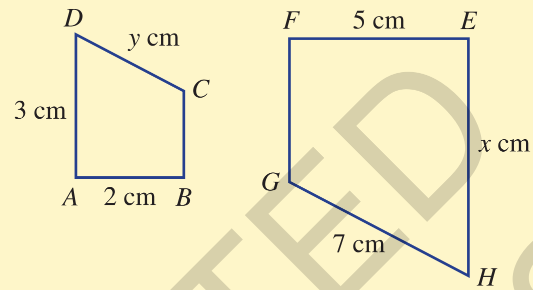
a Write a similarity statement for the two shapes.

b Complete the following: $\frac{EH}{----} = \frac{FG}{----}$.

c Find the scale factor.

d Find the value of x.

e Find the value of y.



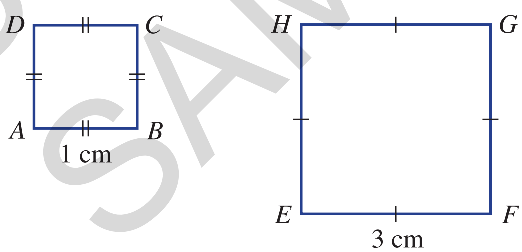
SOLUTION	EXPLANATION
a $ABCD \parallel\!\mid EFGH$ or $ABCD \sim EFGH$	Use the symbol ||| or ~ in similarity statements.
b $\frac{EH}{AD} = \frac{FG}{BC}$	Ensure you match corresponding vertices.
c $\frac{EF}{AB} = \frac{5}{2}$ or 2.5	EF and AB are matching sides and both lengths are given.
d $x = 3 \times 2.5$ $= 7.5$	EH corresponds to AD, which is 3 cm in length. Multiply by the scale factor.
e $y = 7 \div 2.5$ $= 2.8$	DC corresponds to HG, which is 7 cm in length.

Exercise 2D

1–3	3	—

UNDERSTANDING

1 These two figures are squares.



a Would you say that the two squares are similar? Why?

b What is the scale factor when the smaller square is enlarged to the size of the larger square?

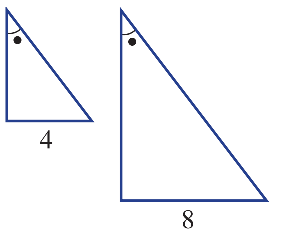
c If the larger square is enlarged by a factor of 5, what would be the side length of the image?

2D

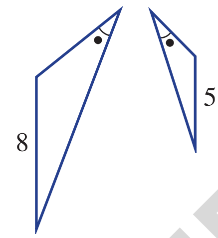
UNDERSTANDING

2 Find the scale factor for each shape and its larger similar image.

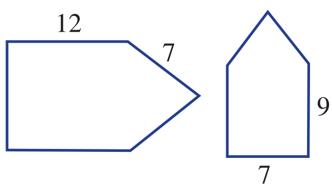
a



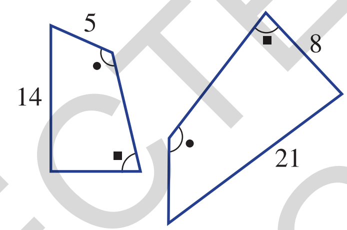
b



c



d



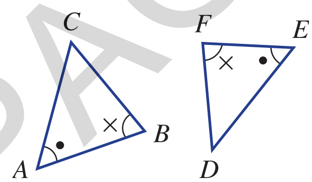
3 The two triangles shown opposite are similar.

a In $\triangle ABC$, which vertex corresponds to (matches) vertex E?

b In $\triangle ABC$, which angle corresponds to $\angle D$?

c In $\triangle DEF$, which side corresponds to BC?

d Write a similarity statement for the two triangles. Write matching vertices in the same order.



4, 5	4–6	5, 6

FLUENCY

Example 7

4 These two shapes are similar.

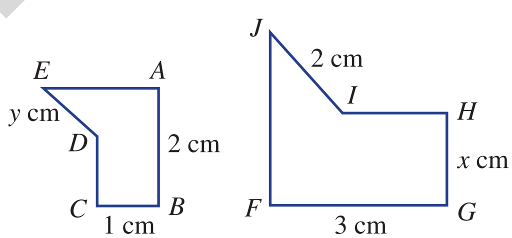
a Write a similarity statement for the two shapes.

b Complete the following: $\dfrac{AB}{----} = \dfrac{DE}{----}$.

c Find the scale factor.

d Find the value of x.

e Find the value of y.



5 These two shapes are similar.

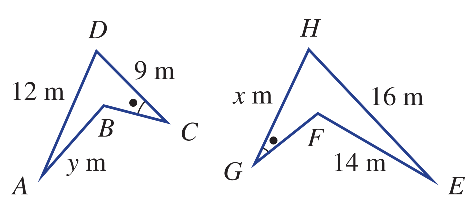
a Write a similarity statement for the two shapes.

b Complete the following: $\dfrac{EF}{----} = \dfrac{----}{CD}$.

c Find the scale factor.

d Find the value of x.

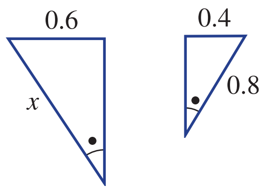
e Find the value of y.

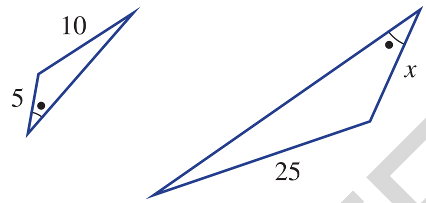


2D

FLUENCY

6 Find the value of the pronumeral in each pair of similar figures. Round to one decimal place where necessary.

a


b


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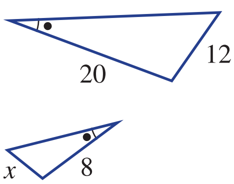

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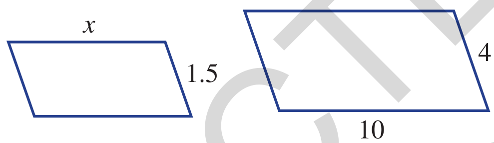
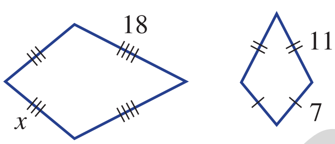
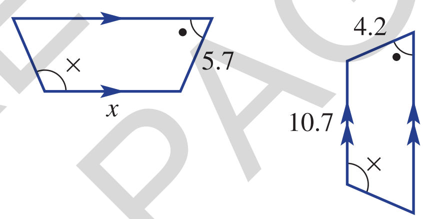


e


f


7, 8	7, 8	8, 9

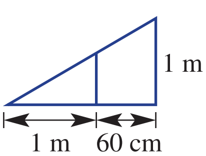
PROBLEM-SOLVING

7 A 50 m tall structure casts a shadow 30 m in length. At the same time, a person casts a shadow of 1.02 m. Estimate the height of the person. (*Hint*: Draw a diagram of two triangles.)

8 A BMX ramp has two vertical supports, as shown.

a Find the scale factor for the two triangles in the diagram.

b Find the length of the inner support.



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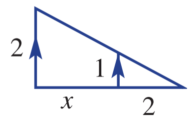
2D

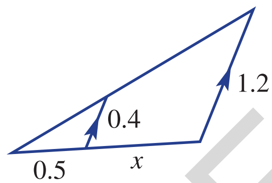
PROBLEM-SOLVING

9 Find the value of the pronumeral if the pairs of triangles are similar. Round to one decimal place in part **d**.

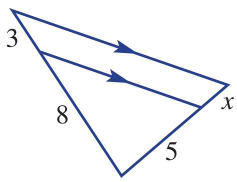
a



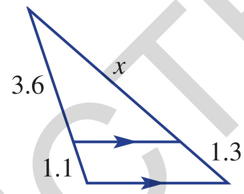
b



c



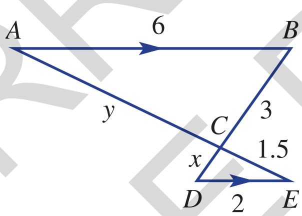
d



10	10, 11	11, 12

REASONING

10 In this diagram, the two triangles are similar and $AB \parallel DE$.



a Which side in $\triangle ABC$ corresponds to DC? Give a reason.

b Write a similarity statement by matching the vertices.

c Find the value of x.

d Find the value of y.

11 Decide whether each statement is true or false.

a All circles are similar.

b All squares are similar.

c All rectangles are similar.

d All rhombuses are similar.

e All parallelograms are similar.

f All trapeziums are similar.

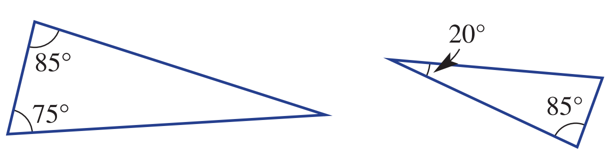
g All kites are similar.

h All isosceles triangles are similar.

i All equilateral triangles are similar.

j All regular hexagons are similar.

12 These two triangles each have two given angles. Decide whether they are similar and give reasons.

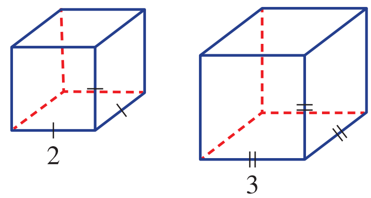


2D

ENRICHMENT

Length, area, volume and self-similarity	—	—	13, 14

13 Shown here is a cube of side length 2 and its image after enlargement.



a Write down the scale factor for the side lengths as an improper fraction.

b Find the area of one face of:

i the smaller cube

ii the larger cube.

c Find the volume of:

i the smaller cube

ii the larger cube.

d Complete this table.

Cube	Length	Area	Volume
Small	2		
Large	3		
Scale factor (fraction)			

e How do the scale factors for Area and Volume relate to the scale factor for side length?

f If the side length scale factor was $\frac{b}{a}$, write down expressions for:

i the area scale factor

ii the volume scale factor.

14 An object that is similar to a part of itself is said to be self-similar. These are objects that, when magnified, reveal the same structural shape. A fern is a good example, as shown in this picture.

Self-similarity is an important area of mathematics, having applications in geography, econometrics and biology.

Use the internet to explore the topic of self-similarity and write a few dot points to summarise your findings.

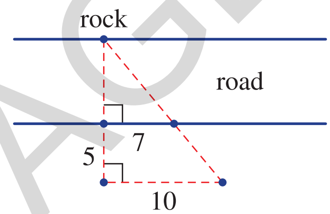
2E Proving similar triangles

As with congruent triangles, there are four tests for proving that two triangles are similar. When solving problems involving similar triangles, it is important to recognise and be able to prove that they are in fact similar.

Similar triangles are used in this bridge structure.

Let's start: How far did the chicken travel?

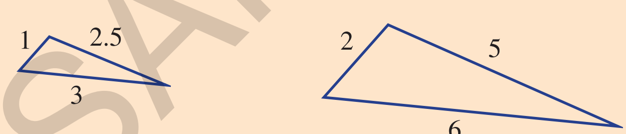
A chicken is considering how far it is across a road, so it places four pebbles in certain positions on one side of the road. Two of the pebbles are directly opposite a rock on the other side of the road. The number of chicken paces between three pairs of the pebbles is shown in the diagram.



- Has the chicken constructed any similar triangles? If so, discuss why they are similar.
- What scale factor is associated with the two triangles?
- Is it possible to find how many paces the chicken must take to get across the road? If so, show a solution.
- Why did the chicken cross the road?
  Answer: To explore similar triangles.

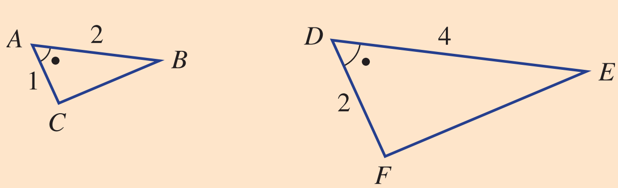
Key ideas

- Two objects are said to be **similar** if they are of the same shape but of different size.
  - For two similar triangles $\triangle ABC$ and $\triangle DEF$, we write $\triangle ABC \ |||\ \triangle DEF$ or $\triangle ABC \sim \triangle DEF$.
  - When comparing two triangles, corresponding sides are opposite equal angles.
- Two triangles can be tested for **similarity** by considering the following conditions.
  - All pairs of corresponding sides are in the same ratio (or proportion) (SSS).



$$\frac{6}{3} = \frac{5}{2.5} = \frac{2}{1} = 2$$

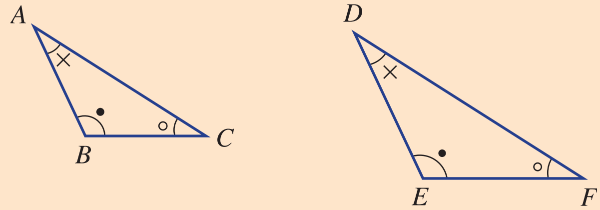
  - Two pairs of corresponding sides are in the same ratio and the included corresponding angles are equal (SAS).



$$\frac{4}{2} = \frac{2}{1} = 2 \text{ and } \angle A = \angle D$$

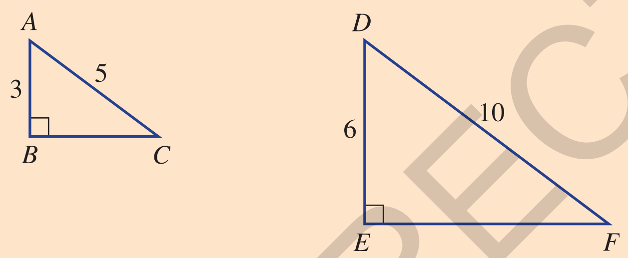
Key ideas

- Three corresponding angles are equal (AAA). (Remember that two pairs of corresponding equal angles implies that all three pairs of corresponding angles are equal.)



$\angle A = \angle D$
$\angle B = \angle E$
$\angle C = \angle F$

- The hypotenuses of two right-angled triangles and another pair of corresponding sides are in the same ratio (RHS).



$\angle B = \angle E = 90^\circ$

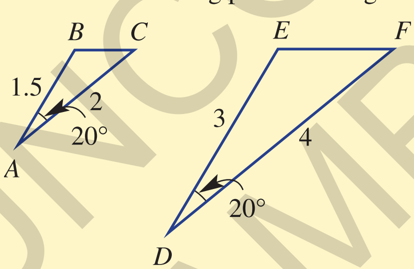
$\frac{10}{5} = \frac{6}{3} = 2$

Note: If the test AAA is not used, then at least two pairs of corresponding sides in the same ratio are required for all the other three tests.

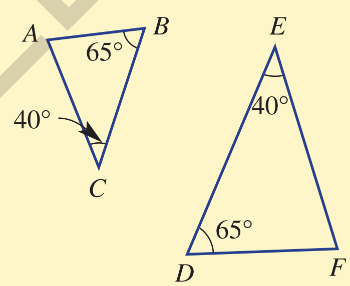
Example 8 Using similarity tests to prove similar triangles

Prove that the following pairs of triangles are similar.

a



b



SOLUTION | **EXPLANATION**

a $\frac{DF}{AC} = \frac{4}{2} = 2$ (ratio of corresponding sides) S

$\frac{DE}{AB} = \frac{3}{1.5} = 2$ (ratio of corresponding sides) S

DF and AC are corresponding sides and DE and AB are corresponding sides, and both pairs are in the same ratio.

$\angle BAC = \angle EDF = 20^\circ$ (given corresponding angles) A

The corresponding angle between the pair of corresponding sides in the same ratio is also equal.

$\therefore\ \triangle ABC \ ||| \ \triangle DEF$ (SAS)

The two triangles are therefore similar.

b $\angle ABC = \angle FDE = 65^\circ$ (given corresponding angles) A
$\angle ACB = \angle FED = 40^\circ$ (given corresponding angles) A
$\therefore\ \triangle ABC \,|||\, \triangle FDE$ (AAA)

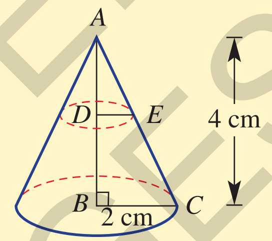
There are two pairs of given corresponding angles. If two pairs of corresponding angles are equal, then the third pair must also be equal (due to angle sum).
The two triangles are therefore similar.

Example 9 Establishing and using similarity

A cone has radius 2 cm and height 4 cm. The top of the cone is cut horizontally through D.

a Prove $\triangle ADE \,|||\, \triangle ABC$.

b If $AD = 1$ cm, find the radius DE.



SOLUTION

a $\angle BAC$ (common) A
$\angle ABC = \angle ADE$ (corresponding angles in parallel lines) A
$\therefore\ \triangle ADE \,|||\, \triangle ABC$ (AAA)

b
$$\frac{DE}{BC} = \frac{AD}{AB}$$
$$\frac{DE}{2} = \frac{1}{4}$$
$$\therefore\ DE = \frac{2}{4}$$
$$= 0.5 \text{ cm}$$

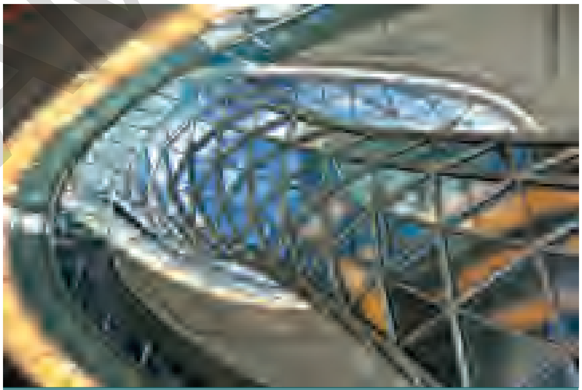
EXPLANATION

All three pairs of corresponding angles are equal.

Therefore, the two triangles are similar.

Given the triangles are similar, the ratio of corresponding sides must be equal.

Solve for DE.



Architecture and engineering are just two fields in which the skills covered in this chapter can be used.

Exercise 2E

1–3	3	—

UNDERSTANDING

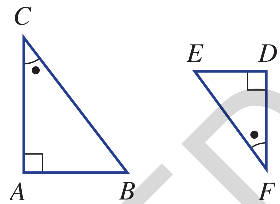
1 This diagram includes two similar triangles.

a Which vertex in $\triangle DEF$ corresponds to vertex B?

b Which angle in $\triangle ABC$ corresponds to $\angle F$?

c Which side in $\triangle ABC$ corresponds to DE?

d Write a similarity statement for the two triangles.



2 This diagram includes two similar triangles.

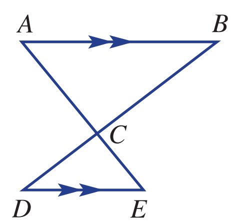
a Which angle in $\triangle CDE$ corresponds to $\angle B$ in $\triangle ABC$ and why?

b Which angle in $\triangle ABC$ corresponds to $\angle E$ in $\triangle CDE$ and why?

c Which angle is vertically opposite $\angle ACB$?

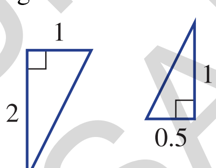
d Which side on $\triangle ABC$ corresponds to side CE on $\triangle CDE$?

e Write a similarity statement, being sure to write matching vertices in the same order.

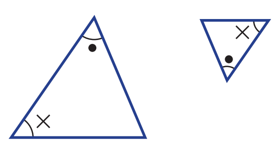


3 Which similarity test (SSS, SAS, AAA or RHS) would be used to prove that these pairs of triangles are similar?

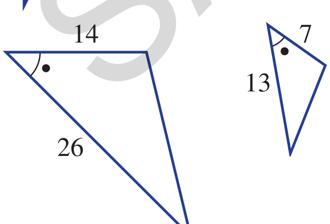
a



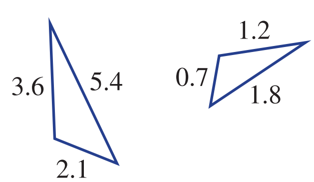
b



c



d



2E

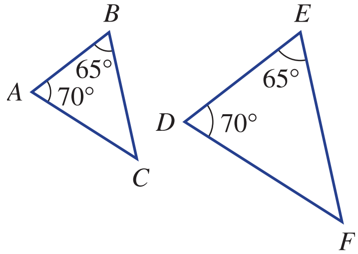
4, 5(½)	4–6(½)	4–6(½)

FLUENCY

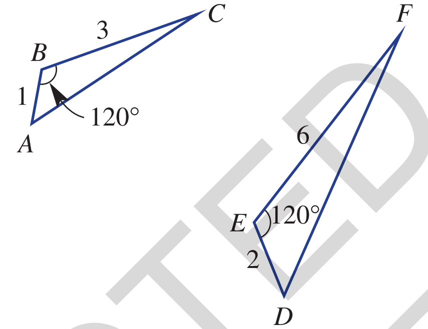
Example 8

4 Prove that the following pairs of triangles are similar.

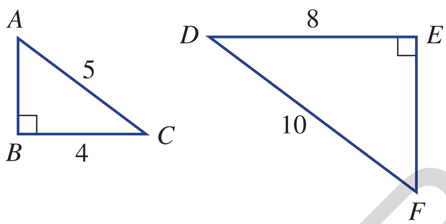
a



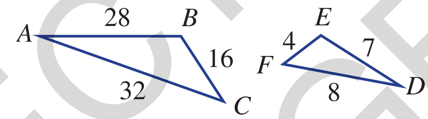
b



c

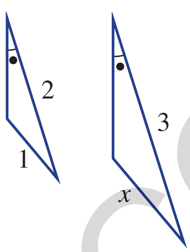


d

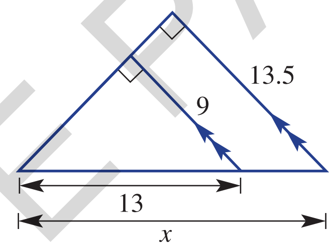


5 Find the value of the pronumerals in these pairs of similar triangles.

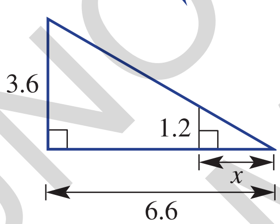
a



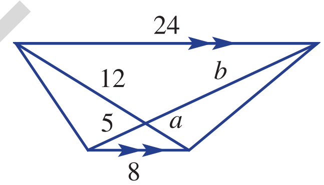
b



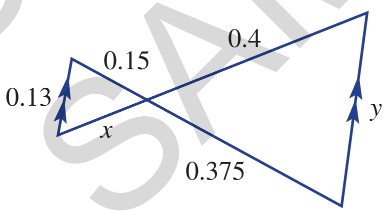
c



d



e



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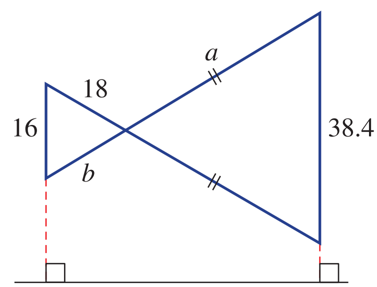
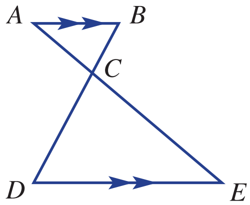


2E

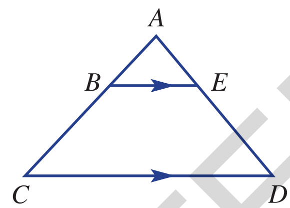
FLUENCY

6 For the following proofs, give reasons at each step.

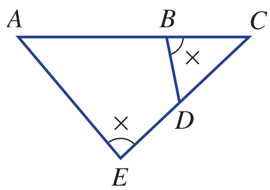
a Prove $\triangle ABC \;|||\; \triangle EDC$.



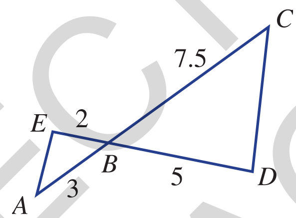
b Prove $\triangle ABE \;|||\; \triangle ACD$.



c Prove $\triangle BCD \;|||\; \triangle ECA$.



d Prove $\triangle AEB \;|||\; \triangle CDB$.



7, 8	8, 9	9, 10

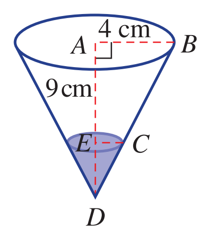
PROBLEM-SOLVING

Example 9

7 A right cone with radius 4 cm has a total height of 9 cm. It contains an amount of water, as shown.

a Prove $\triangle EDC \;|||\; \triangle ADB$.

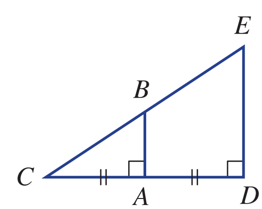
b If the depth of water in the cone is 3 cm, find the radius of the water surface in the cone.



8 A ramp is supported by a vertical stud AB, where A is the midpoint of CD. It is known that $CD = 4$ m and that the ramp is 2.5 m high; i.e. $DE = 2.5$ m.

a Prove $\triangle BAC \;|||\; \triangle EDC$.

b Find the length of the stud AB.

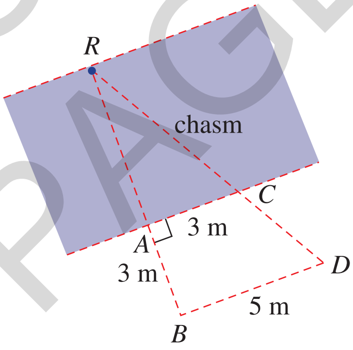


2E

PROBLEM-SOLVING

9 At a particular time in the day, Aaron casts a shadow 1.3 m long and Jack, who is 1.75 m tall, casts a shadow 1.2 m long. Find Aaron's height, correct to two decimal places.

10 To determine the width of a chasm, a marker (*A*) is placed directly opposite a rock (*R*) on the other side. Point *B* is placed 3 m away from point *A*, as shown. Marker *C* is placed 3 m along the edge of the chasm, and marker *D* is placed so that *BD* is parallel to *AC*. Markers *C* and *D* and the rock are collinear (i.e. lie in a straight line). If *BD* measures 5 m, find the width of the chasm (*AR*).

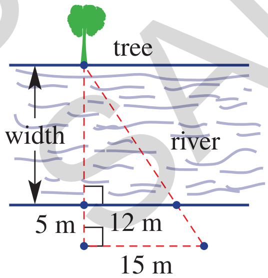


11	11, 12	11–13

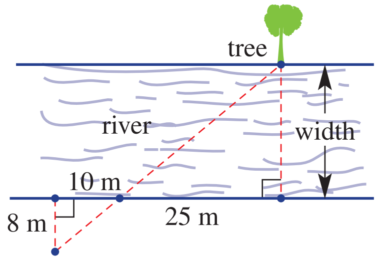
REASONING

11 Aiden and Jenny come to a river and notice a tree on the opposite bank. Separately they decide to place rocks (indicated with dots) on their side of the river to try to calculate the river's width. They then measure the distances between some pairs of rocks, as shown.

Aiden's rock placement



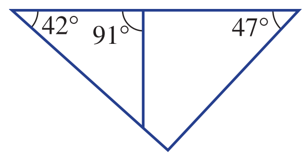
Jenny's rock placement



a Have both Aiden and Jenny constructed a pair of similar triangles? Give reasons.
b Use Jenny's triangles to calculate the width of the river.
c Use Aiden's triangles to calculate the width of the river.
d Which pair of triangles did you prefer to use? Give reasons.

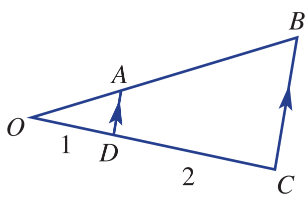
REASONING

12 There are two triangles in this diagram, each showing two given angles. Explain why they are similar.

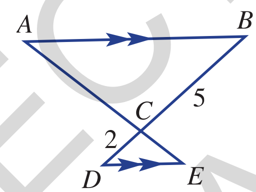


13 Prove the following, giving reasons.

a $OB = 3OA$



b $AE = \frac{7}{5}AC$



Proving Pythagoras' theorem	—	—	14

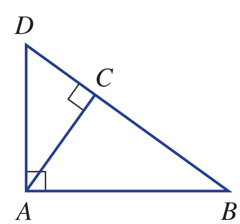
ENRICHMENT

14 In this figure $\triangle ABD$, $\triangle CBA$ and $\triangle CAD$ are right angled.

a Prove $\triangle ABD \mathrel{|||} \triangle CBA$. Hence, prove $AB^2 = CB \times BD$.

b Prove $\triangle ABD \mathrel{|||} \triangle CAD$. Hence, prove $AD^2 = CD \times BD$.

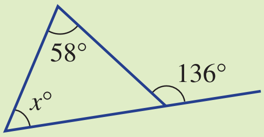
c Hence, prove Pythagoras' theorem $AB^2 + AD^2 = BD^2$.



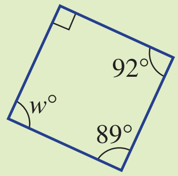
Progress quiz

2A **1** Find the size of each pronumeral in the following polygons, giving reasons.

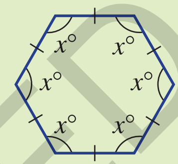
a



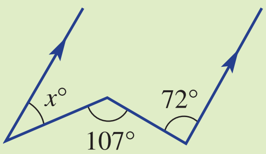
b



c



d



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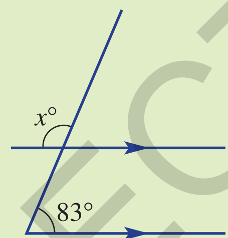

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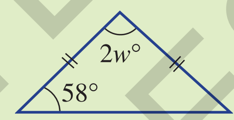
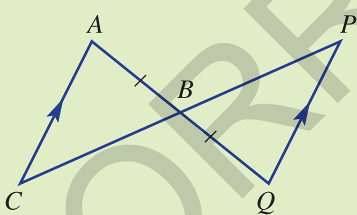


2B **2** **a** Prove that $\triangle ABC$ is congruent to $\triangle QBP$.



b Prove that B is the midpoint of CP.

2C **3** Prove that a rhombus has its diagonals perpendicular to each other.

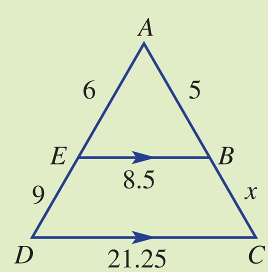
2C **4** Prove that if the opposite sides of a quadrilateral are parallel, then the quadrilateral is a parallelogram and the opposite sides are equal in length.

2D **5** For the two similar triangles shown:

a Write a similarity statement.

b Find the scale factor.

c Find the value of x.



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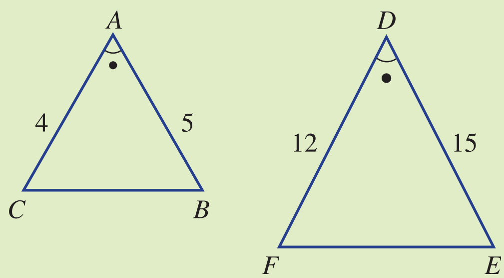
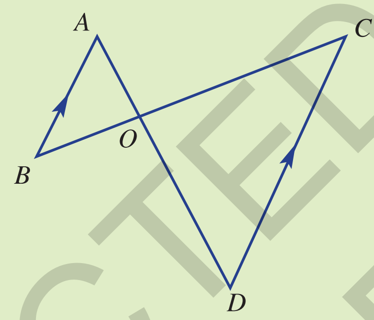
2E **6** Prove that the following pairs of triangles are similar.

a



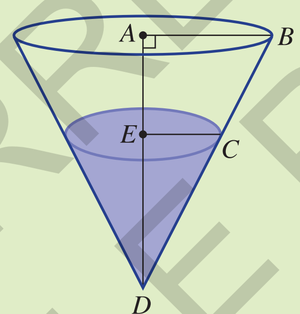
b



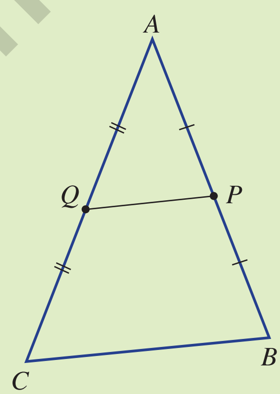
2E **7** A cone with radius 6 cm and height 10 cm is filled with water to a height of 5 cm.

a Prove that $\triangle EDC$ is similar to $\triangle ADB$.

b Find the radius of the water's surface (EC).



2E **8** Prove that if the midpoints, Q and P, of two sides of a triangle ABC are joined as shown, then QP is $\frac{1}{2}$ that of CB.



2F Circles and chord properties 10A

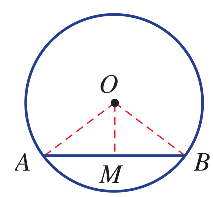
Interactive
Widgets
HOTsheets
Walkthroughs

Although a circle appears to be a very simple object, it has many interesting geometrical properties. In this section we look at radii and chords in circles, and then explore and apply the properties of these objects. We use congruence to prove many of these properties.

Let's start: Dynamic chords

This activity would be enhanced with the use of computer dynamic geometry.

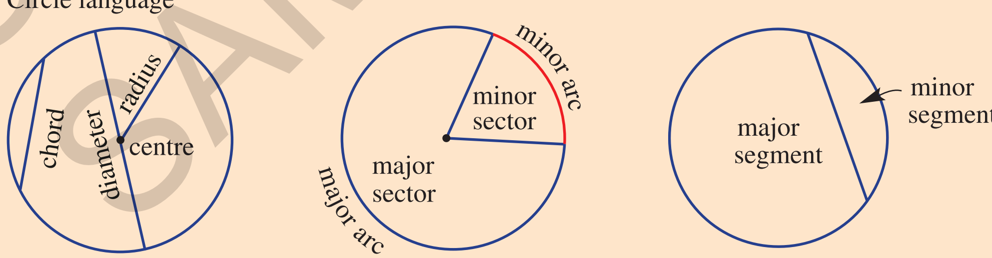
Chord AB sits on a circle with centre O. M is the midpoint of chord AB. Explore with dynamic geometry software or discuss the following.



- Is $\triangle OAB$ isosceles and if so why?
- Is $\triangle OAM \equiv \triangle OBM$ and if so why?
- Is $AB \perp OM$ and if so why?
- Is $\angle AOM = \angle BOM$ and if so why?

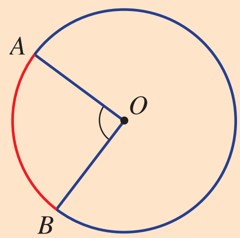
Key ideas

■ Circle language

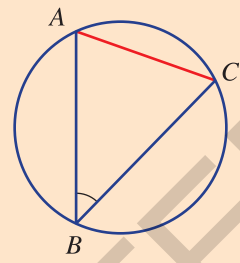


Key ideas

- An angle is **subtended** by an arc or chord if the arms of the angle meet the endpoints of the arc or chord.

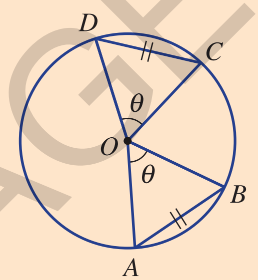


$\angle AOB$ is subtended at the centre by the minor arc AB.

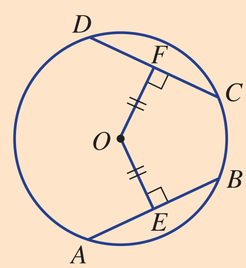


$\angle ABC$ is subtended at the circumference by the chord AC.

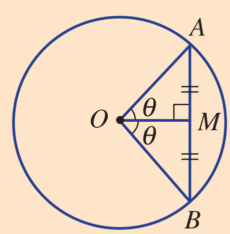
- **Chord theorem 1**: Chords of equal length subtend equal angles at the centre of the circle.
  - If $AB = CD$, then $\angle AOB = \angle COD$.
  - Conversely, if chords subtend equal angles at the centre of the circle, then the chords are of equal length.



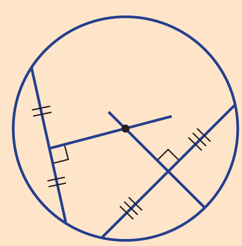
- **Chord theorem 2**: Chords of equal length are equidistant (i.e. of equal distance) from the centre of the circle.
  - If $AB = CD$, then $OE = OF$.
  - Conversely, if chords are equidistant from the centre of the circle, then the chords are of equal length.



- **Chord theorem 3**: The perpendicular from the centre of the circle to the chord bisects the chord and the angle at the centre subtended by the chord.
  - If $OM \perp AB$, then $AM = BM$ and $\angle AOM = \angle BOM$.
  - Conversely, if a radius bisects the chord (or angle at the centre subtended by the chord), then the radius is perpendicular to the chord.



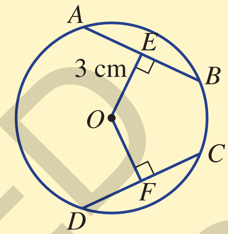
- **Chord theorem 4**: The perpendicular bisectors of every chord of a circle intersect at the centre of the circle.
  - Constructing perpendicular bisectors of two chords will therefore locate the centre of a circle.



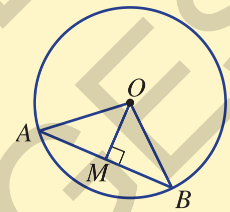
Example 10 Using chord theorems

For each part, use the given information and state which chord theorem is used.

a Given $AB = CD$ and $OE = 3$ cm, find OF.



b Given $OM \perp AB$, $AB = 10$ cm and $\angle AOB = 92°$, find AM and $\angle AOM$.

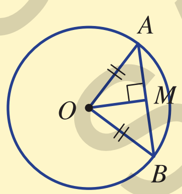


SOLUTION	EXPLANATION
a $OF = 3$ cm (using chord theorem 2)	Chords of equal length are equidistant from the centre.
b Using chord theorem 3: $AM = 5$ cm $\angle AOM = 46°$	The perpendicular from the centre to the chord bisects the chord and the angle at the centre subtended by the chord. $10 \div 2 = 5$ and $92 \div 2 = 46$.

Example 11 Proving chord theorems

Prove chord theorem 3 in that the perpendicular from the centre of the circle to the chord bisects the chord and the angle at the centre subtended by the chord.

SOLUTION



EXPLANATION

First, draw a diagram to represent the situation. The perpendicular forms a pair of congruent triangles.

$\angle OMA = \angle OMB = 90°$ (given) R

$OA = OB$ (both radii) H

OM (common) S

$\therefore \triangle OMA \equiv \triangle OMB$ (RHS)

$\therefore AM = BM$ and $\angle AOM = \angle BOM$

Corresponding sides and angles in congruent triangles are equal.

Exercise 2F

1–4	3, 4	—

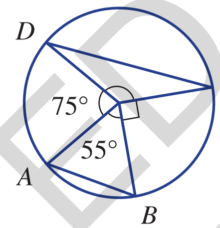
UNDERSTANDING

1 Construct a large circle, then draw and label these features.

a a chord **b** radius **c** centre **d** minor sector **e** major sector

2 Give the size of the angle in this circle subtended by the following.

a chord AB **b** minor arc BC
c minor arc AD **d** chord DC



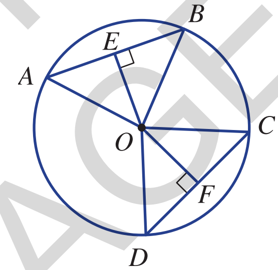
3 In this circle chords AB and CD are of equal length.

a Measure $\angle AOB$ and $\angle COD$.

b What do you notice about your answers from part **a**? Which chord theorem does this relate to?

c Measure OE and OF.

d What do you notice about your answers from part **c**? Which chord theorem does this relate to?

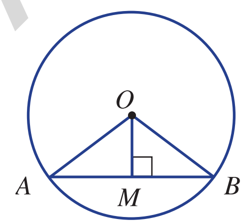


4 In this circle $OM \perp AB$.

a Measure AM and BM.

b Measure $\angle AOM$ and $\angle BOM$.

c What do you notice about your answers from parts **a** and **b**? Which chord theorem does this relate to?



5, 6	5, 7(½)	5, 6, 7(½)

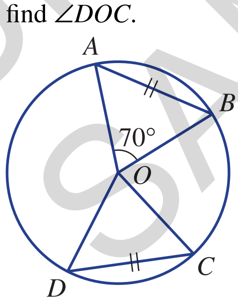
FLUENCY

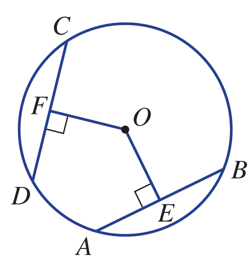
Example 10

5 For each part, use the information given and state which chord theorem is used.

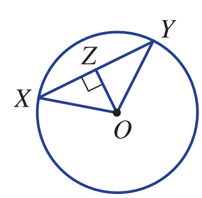
a Given $AB = CD$ and $\angle AOB = 70°$, find $\angle DOC$.

b Given $AB = CD$ and $OF = 7.2$ cm, find OE.





c Given $OZ \perp XY$, $XY = 8$ cm and $\angle XOY = 102°$, find XZ and $\angle XOZ$.



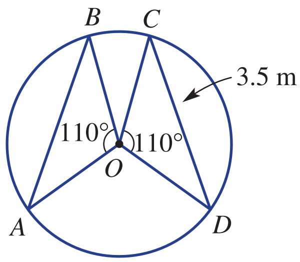
2F

FLUENCY

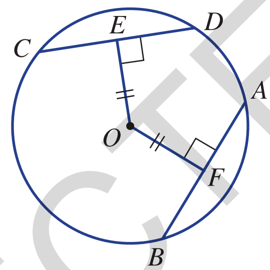
6 The perpendicular bisectors of two different chords of a circle are constructed. Describe where they intersect.

7 Use the information given to complete the following.

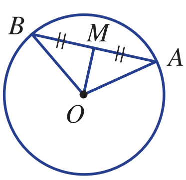
a Given $\angle AOB = \angle COD$ and $CD = 3.5$ m, find AB.



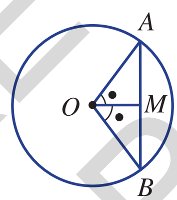
b Given $OE = OF$ and $AB = 9$ m, find CD.



c Given M is the midpoint of AB, find $\angle OMB$.



d Given $\angle AOM = \angle BOM$, find $\angle OMB$.

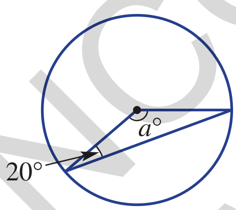


8	8(½), 9	9, 10

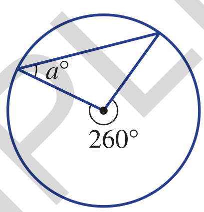
PROBLEM-SOLVING

8 Find the size of each unknown angle $a°$.

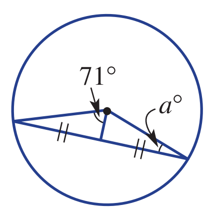
a



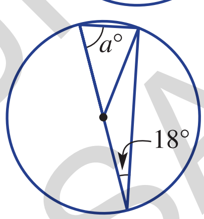
b



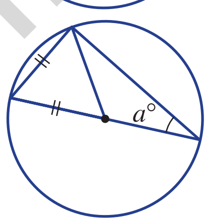
c



d



e



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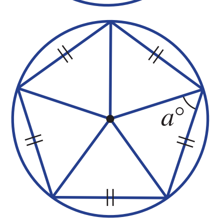
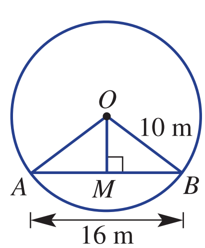


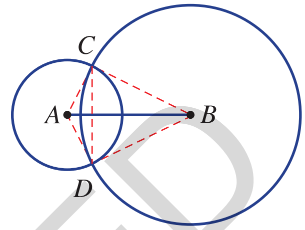
9 Find the length OM.
(*Hint*: Use Pythagoras' theorem.)



2F

PROBLEM-SOLVING

10 In this diagram, radius $AD = 5$ mm, radius $BD = 12$ mm and chord $CD = 8$ mm. Find the exact length of AB, in surd form.



11	11, 12	12, 13

REASONING

Example 11

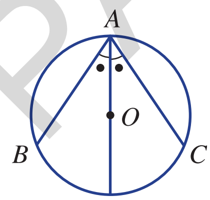
11 **a** Prove chord theorem 1 in that chords of equal length subtend equal angles at the centre of the circle.

b Prove the converse of chord theorem 1 in that if chords subtend equal angles at the centre of the circle then the chords are of equal length.

12 **a** Prove that if a radius bisects a chord of a circle then the radius is perpendicular to the chord.

b Prove that if a radius bisects the angle at the centre subtended by the chord, then the radius is perpendicular to the chord.

13 In this circle $\angle BAO = \angle CAO$. Prove $AB = AC$.
(*Hint*: Construct two triangles.)



Common chord proof

—	—	14

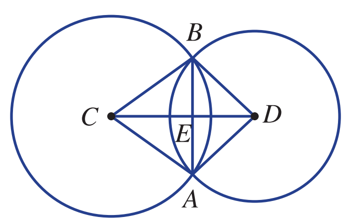
ENRICHMENT

14 For this diagram, prove $CD \perp AB$ by following these steps.

a Prove $\triangle ACD \equiv \triangle BCD$.

b Hence, prove $\triangle ACE \equiv \triangle BCE$.

c Hence, prove $CD \perp AB$.

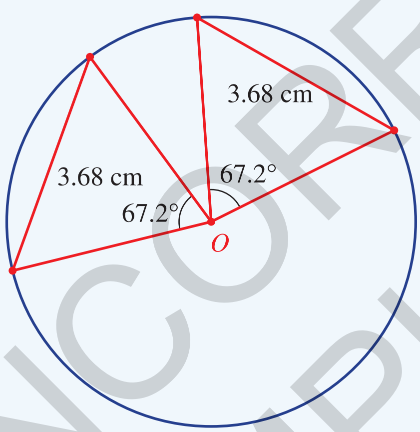


Exploring chord theorems with dynamic geometry

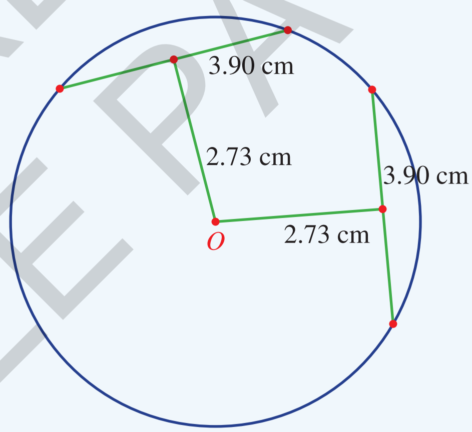
1. Construct a circle with centre O and any given radius.
2. Construct chords of equal length by rotating a chord about the centre by a given angle.
3. Illustrate the four chord properties by constructing line segments, as shown below. Measure corresponding angles and lengths to illustrate the chord properties.
   - Chord theorem 1: Chords of equal length subtend equal angles at the centre of the circle.
   - Chord theorem 2: Chords of equal length are equidistant from the centre of the circle.
   - Chord theorem 3: The perpendicular from the centre of the circle to the chord bisects the chord and the angle at the centre subtended by the chord.
   - Chord theorem 4: The perpendiculars of every chord of a circle intersect at the centre of the circle.
4. Drag the circle or one of the points on the circle to check that the properties are retained.

Chord theorem illustrations

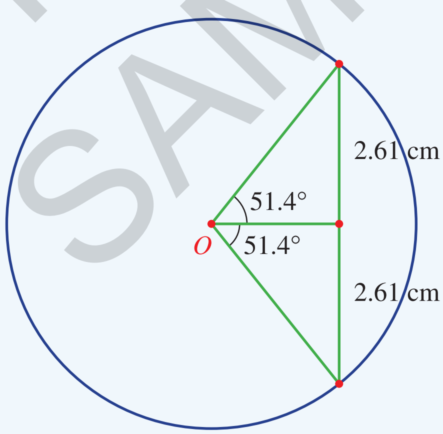
Chord theorem 1



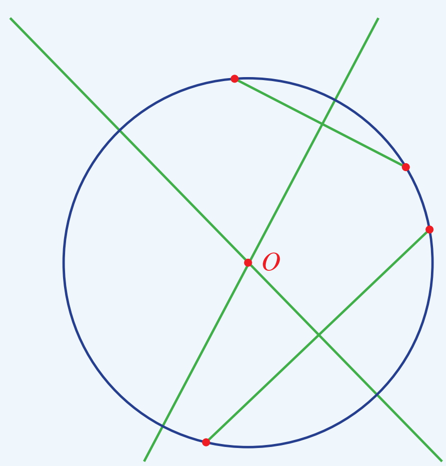
Chord theorem 2



Chord theorem 3



Chord theorem 4



2G Angle properties of circles – theorems 1 and 2 10A

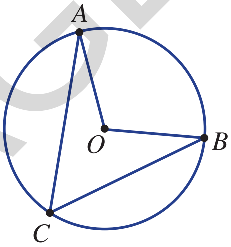
Interactive
Widgets
HOTsheets
Walkthroughs

The special properties of circles extend to the pairs of angles formed by radii and chords intersecting at the circumference. In this section we explore the relationship between angles at the centre and at the circumference subtended by the same arc.

Let's start: Discover angle properties – theorems 1 and 2

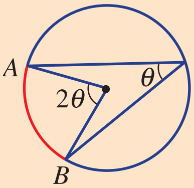
This activity can be completed with the use of a protractor and pair of compasses, but would be enhanced by using computer dynamic geometry software.

- First, construct a circle and include two radii and two chords, as shown. The size of the circle and position of points A, B and C on the circumference can vary.
- Measure $\angle ACB$ and $\angle AOB$. What do you notice?
- Now construct a new circle with points A, B and C at different points on the circumference. (If dynamic software is used simply drag the points.) Measure $\angle ACB$ and $\angle AOB$ once again. What do you notice?
- Construct a new circle with $\angle AOB = 180^\circ$ so AB is a diameter. What do you notice about $\angle ACB$?

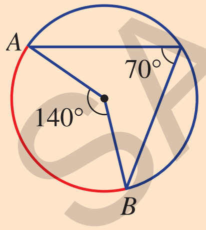


Key ideas

- **Circle theorem 1:** Angles at the centre and circumference
  - The angle at the centre of a circle is twice the angle at a point on the circle subtended by the same arc.



For example:

1


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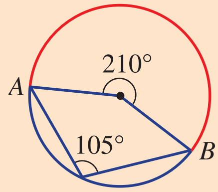

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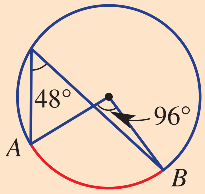
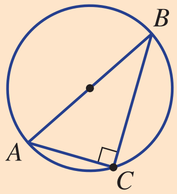


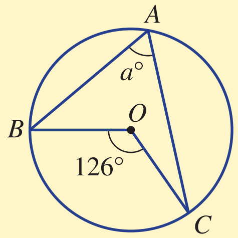
- **Circle theorem 2:** Angle in a semicircle
  - The angle in a semicircle is 90°.
    This is a specific case of theorem 1, where $\angle ACB$ is known as the angle in a semicircle.



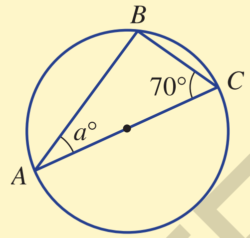
Example 12 Applying circle theorems 1 and 2

Find the value of the pronumerals in these circles.

a



b



SOLUTION

a $2a = 126$

$\therefore\ a = 63$

b $\angle ABC$ is 90°.

$\therefore\ a + 90 + 70 = 180$

$\therefore\ a = 20$

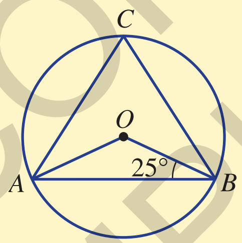
EXPLANATION

From circle theorem 1, $\angle BOC = 2\angle BAC$.

AC is a diameter, and from circle theorem 2 $\angle ABC = 90°$.

Example 13 Combining circle theorems with other circle properties

Find the size of $\angle ACB$.



SOLUTION

$\angle OAB = 25°$

$\angle AOB = 180° - 2 \times 25°$

$= 130°$

$\therefore\ \angle ACB = 65°$

EXPLANATION

$\triangle AOB$ is isosceles.

Angle sum of a triangle is 180°.

The angle at the circumference is half the angle at the centre subtended by the same arc.

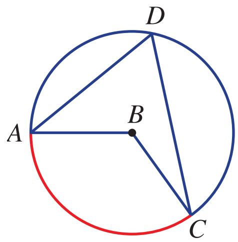
Exercise 2G

1–3	2, 3	—

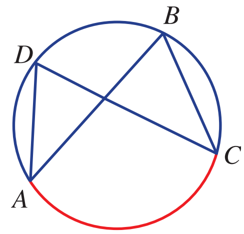
UNDERSTANDING

1 Name another angle that is subtended by the same arc as $\angle ABC$ in these circles.

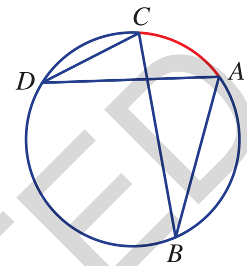
a



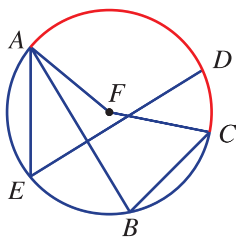
b



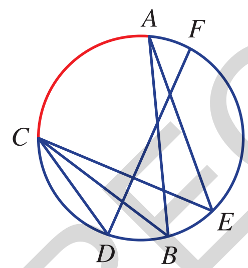
c



d



e



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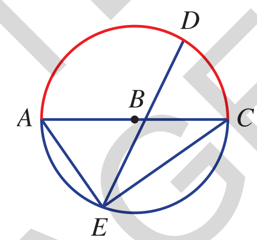
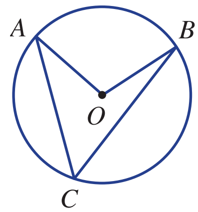


2 For this circle, O is the centre.



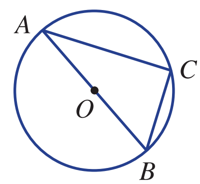
a Name the angle at the centre of the circle.

b Name the angle at the circumference of the circle.

c If $\angle ACB = 40^\circ$, find $\angle AOB$ using circle theorem 1.

d If $\angle AOB = 122^\circ$, find $\angle ACB$ using circle theorem 1.

3 For this circle AB is a diameter.



a What is the size of $\angle AOB$?

b What is the size of $\angle ACB$ using circle theorem 2?

c If $\angle CAB = 30^\circ$, find $\angle ABC$.

d If $\angle ABC = 83^\circ$, find $\angle CAB$.

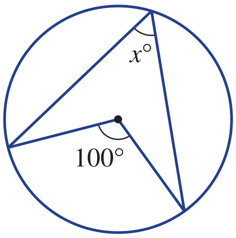
2G

4–6(½)	4–7(½)	4–7(½)

FLUENCY

Example 12a

4 Find the value of x° in these circles.

a


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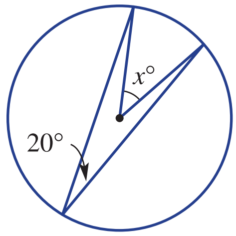

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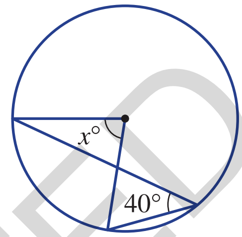

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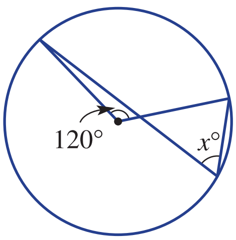

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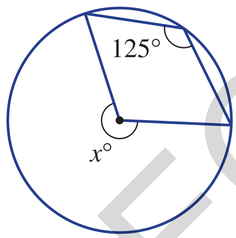

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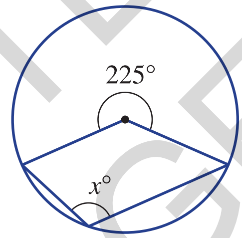

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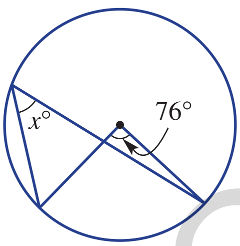

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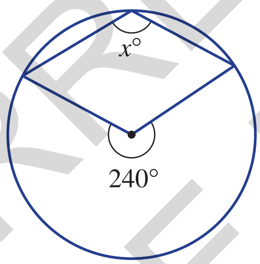

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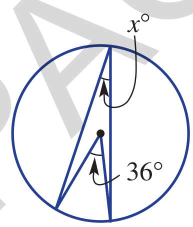
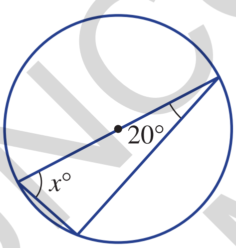


Example 12b

5 Find the value of x° in these circles.

a


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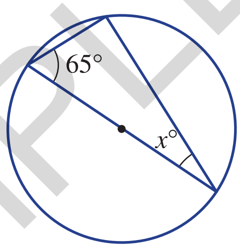

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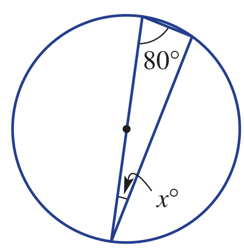
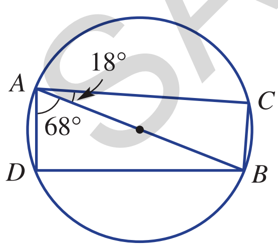


6 Find the size of both $\angle ABC$ and $\angle ABD$.

a


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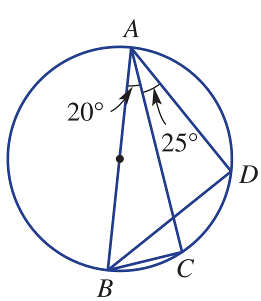

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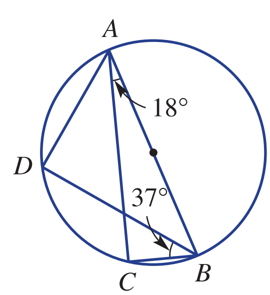
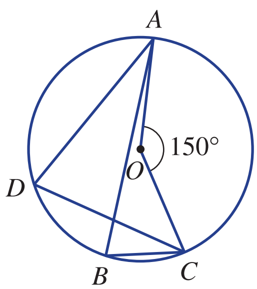


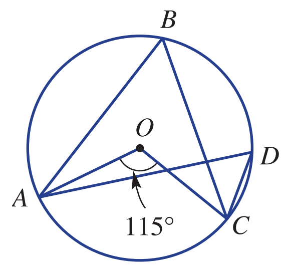
2G

FLUENCY

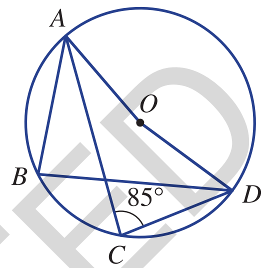
7 **a** Find $\angle ADC$ and $\angle ABC$.



b Find $\angle ABC$ and $\angle ADC$.



c Find $\angle AOD$ and $\angle ABD$.

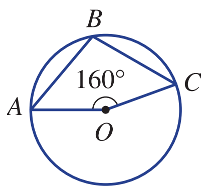


8	8–9(½)	8–9(½)

PROBLEM-SOLVING

8 Find $\angle ABC$.

a



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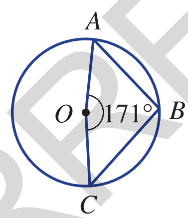

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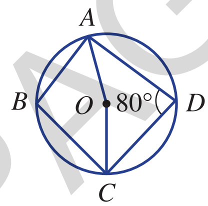

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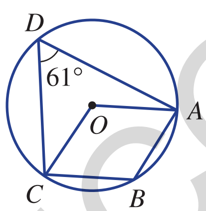

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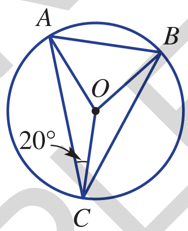

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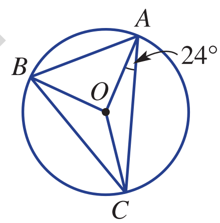
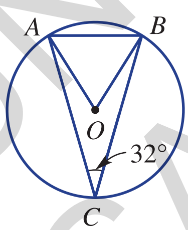


Example 13

9 Find $\angle OAB$.

a



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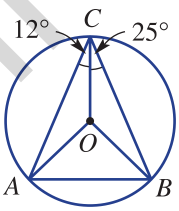

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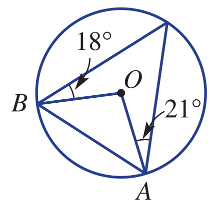

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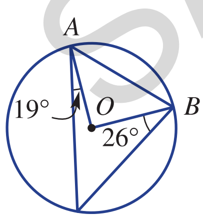

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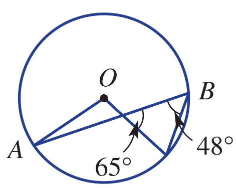

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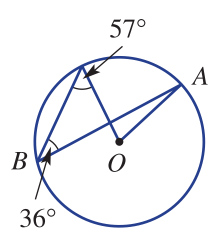
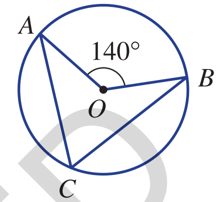


2G

10	10, 11	11–13

REASONING

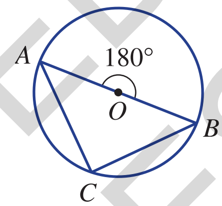
10 **a** For the first circle shown, use circle theorem 1 to find $\angle ACB$.



b For the second circle shown, use circle theorem 1 to find $\angle ACB$.

c For the second circle what does circle theorem 2 say about $\angle ACB$?

d Explain why circle theorem 2 can be thought of as a special case of circle theorem 1.

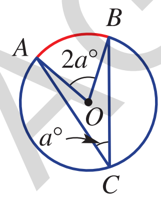


11 These two circles illustrate circle theorem 1 for both a minor arc and a major arc.

a When a minor arc is used, answer true or false.

- **i** $\angle AOB$ is always acute.
- **ii** $\angle AOB$ can be acute or obtuse.
- **iii** $\angle ACB$ is always acute.
- **iv** $\angle ACB$ can be acute or obtuse.

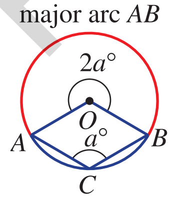
minor arc *AB*



b When a major arc is used, answer true or false.

- **i** $\angle ACB$ can be acute.
- **ii** $\angle ACB$ is always obtuse.
- **iii** The angle at the centre $(2a^\circ)$ is a reflex angle.
- **iv** The angle at the centre $(2a^\circ)$ can be obtuse.

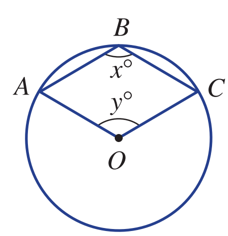
major arc *AB*



12 Consider this circle.

a Write reflex $\angle AOC$ in terms of x.

b Write y in terms of x.



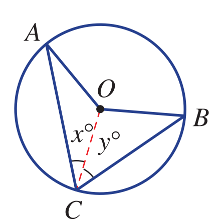
13 Prove circle theorem 1 for the case illustrated in this circle by following these steps and letting $\angle OCA = x^\circ$ and $\angle OCB = y^\circ$.

a Find $\angle AOC$ in terms of x, giving reasons.

b Find $\angle BOC$ in terms of y, giving reasons.

c Find $\angle AOB$ in terms of x and y.

d Explain why $\angle AOB = 2\angle ACB$.

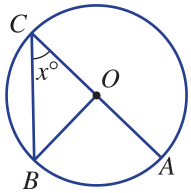


Proving all cases | — | — | 14, 15 |

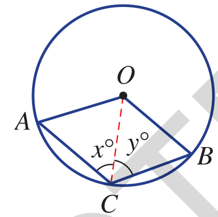
ENRICHMENT

14 Question **13** sets out a proof for circle theorem 1 using a given illustration. Now use a similar technique for these cases.

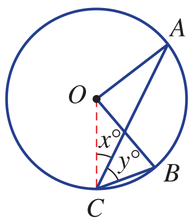
a Prove $\angle AOB = 2\angle ACB$;
i.e. prove $\angle AOB = 2x^\circ$.



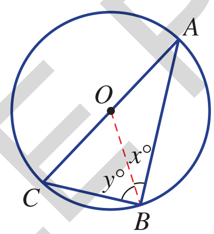
b Prove reflex $\angle AOB = 2\angle ACB$;
i.e. prove reflex $\angle AOB = 2(x + y)^\circ$.



c Prove $\angle AOB = 2\angle ACB$;
i.e. prove $\angle AOB = 2y^\circ$.



15 Prove circle theorem 2 by showing that $x + y = 90$.



Exploring circle theorems with dynamic geometry software

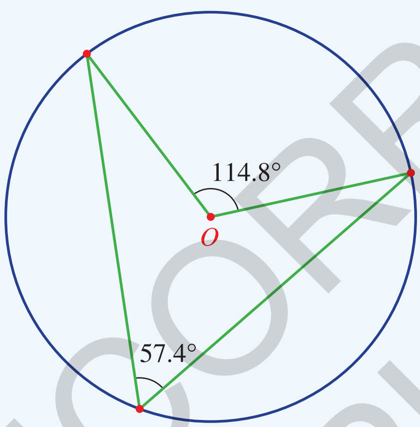
1 Construct a circle with centre O and any given radius.

2 Illustrate the four angle properties by constructing line segments, as shown.
Measure corresponding angles to illustrate the angle properties.
- Circle theorem 1: The angle at the centre of a circle is twice the angle at a point on the circle subtended by the same arc.
- Circle theorem 2: The angle in a semicircle is 90°.
- Circle theorem 3: Angles at the circumference of a circle subtended by the same arc are equal.
- Circle theorem 4: Opposite angles in a cyclic quadrilateral are supplementary.

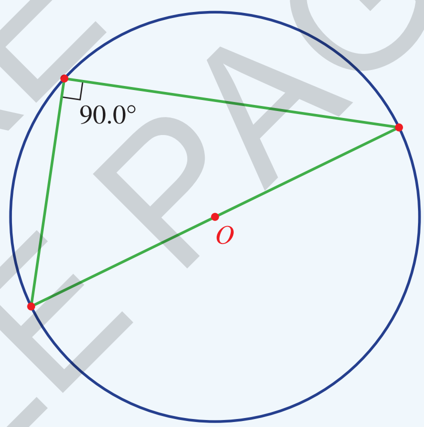
3 Drag the circle or one of the points on the circle to check that the properties are retained.

Circle theorem illustrations

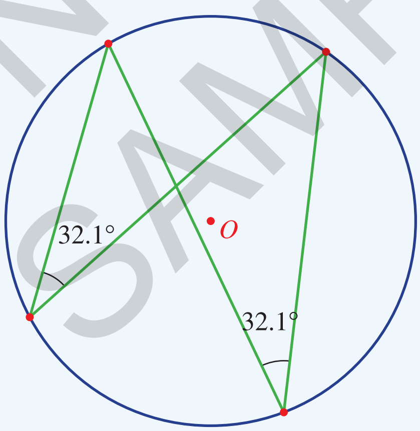
Circle theorem 1



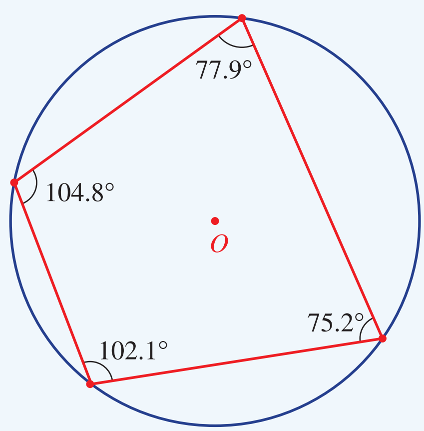
Circle theorem 2



Circle theorem 3



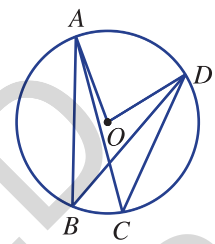
Circle theorem 4



2H Angle properties of circles – theorems 3 and 4 10A

Interactive
Widgets
HOTsheets
Walkthroughs

When both angles are at the circumference, there are two more important properties of pairs of angles in a circle to consider.



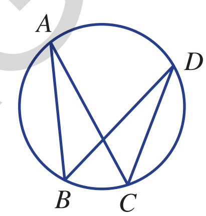
You will recall from circle theorem 1 that in this circle $\angle AOD = 2\angle ABD$ and also $\angle AOD = 2\angle ACD$. This implies that $\angle ABD = \angle ACD$, which is an illustration of circle theorem 3 – angles at the circumference subtended by the same arc are equal.

The fourth theorem relates to cyclic quadrilaterals, which have all four vertices sitting on the same circle. This also will be explored in this section.

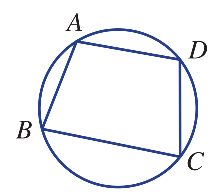
Let's start: Discover angle properties – theorems 3 and 4

Once again, use a protractor and a pair of compasses for this exercise or use computer dynamic geometry software.

- Construct a circle with four points at the circumference, as shown.
- Measure $\angle ABD$ and $\angle ACD$. What do you notice? Drag A, B, C or D and compare the angles.



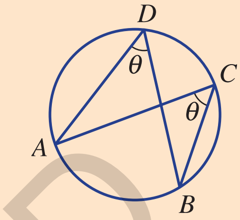
- Now construct this cyclic quadrilateral (or drag point C if using dynamic geometry software).
- Measure $\angle ABC$, $\angle BCD$, $\angle CDA$ and $\angle DAB$. What do you notice? Drag A, B, C or D and compare angles.



The study of angle properties of circles by ancient Greek mathematicians laid the foundations of trigonometry. Much of the work was done by Ptolemy, who lived in the city of Alexandria, Egypt. The photo shows ancient monuments that existed in Ptolemy's time and are still standing in Alexandria today.

Key ideas

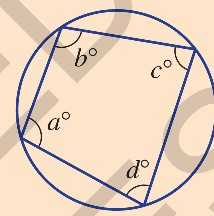
- **Circle theorem 3**: Angles at the circumference
  - Angles at the circumference of a circle subtended by the same arc are equal.
    - As shown in the diagram, $\angle C = \angle D$ but note also that $\angle A = \angle B$.



- **Circle theorem 4**: Opposite angles in cyclic quadrilaterals
  - Opposite angles in a **cyclic quadrilateral** are supplementary (sum to 180°). A cyclic quadrilateral has all four vertices sitting on the same circle.

$a + c = 180$

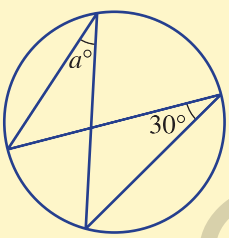
$b + d = 180$



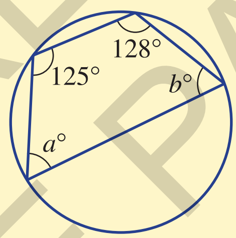
Example 14 Applying circle theorems 3 and 4

Find the value of the pronumerals in these circles.

a



b



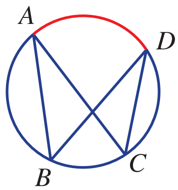
SOLUTION	EXPLANATION
a $a = 30$	The $a°$ and 30° angles are subtended by the same arc. This is an illustration of circle theorem 3.
b $a + 128 = 180$ $\therefore\ a = 52$ $b + 125 = 180$ $\therefore\ b = 55$	The quadrilateral is cyclic, so opposite angles sum to 180°. This is an illustration of circle theorem 4.

Exercise 2H

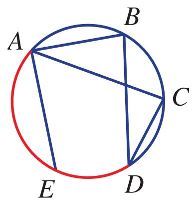
UNDERSTANDING		
1–3	2, 3	—

1 Name another angle that is subtended by the same arc as $\angle ABD$.

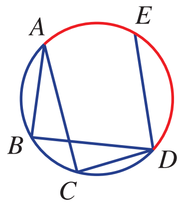
a



b



c



2H

UNDERSTANDING

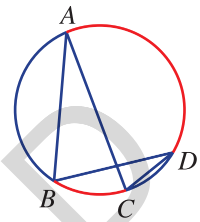
2 For this circle, answer the following.

a Name two angles subtended by arc AD.

b Using circle theorem 3, state the size of $\angle ACD$ if $\angle ABD = 85°$.

c Name two angles subtended by arc BC.

d Using circle theorem 3, state the size of $\angle BAC$ if $\angle BDC = 17°$.



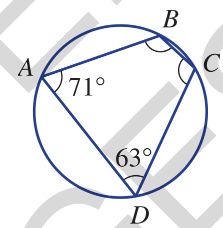
3 Circle theorem 4 states that opposite angles in a cyclic quadrilateral are supplementary.

a What does it mean when we say two angles are supplementary?

b Find $\angle ABC$.

c Find $\angle BCD$.

d Check that $\angle ABC + \angle BCD + \angle CDA + \angle DAB = 360°$.



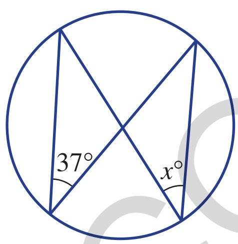
4–5(½)	4–5(½)	4–5(½)

FLUENCY

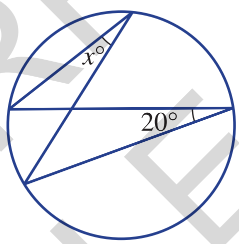
Example 14a

4 Find the value of x in these circles.

a



b



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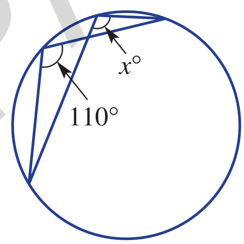
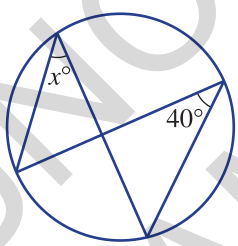


d



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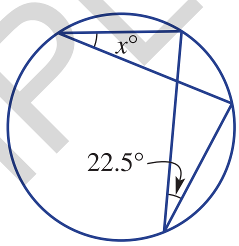

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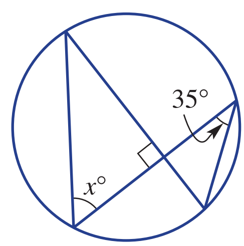
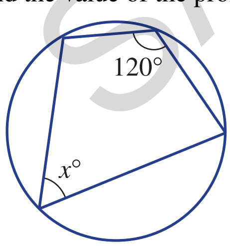


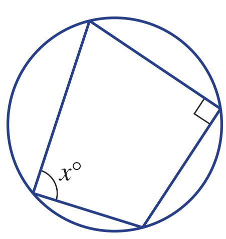
Example 14b

5 Find the value of the pronumerals in these circles.

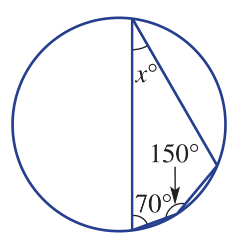
a



b

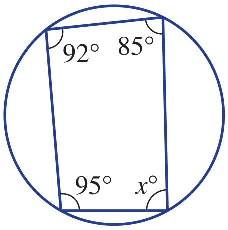


c

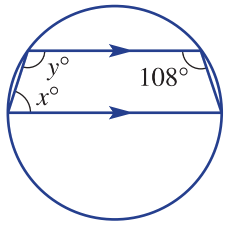


FLUENCY

d



e



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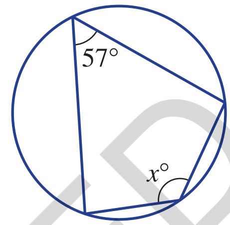
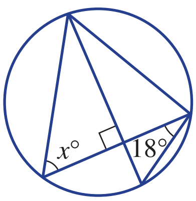


6	6–7(½)	6–7(½)

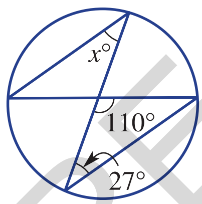
PROBLEM-SOLVING

6 Find the value of $x°$.

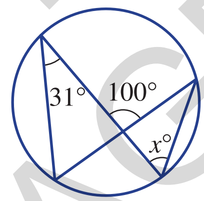
a



b



c



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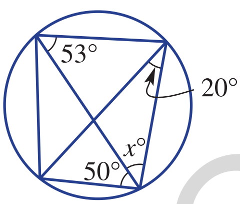

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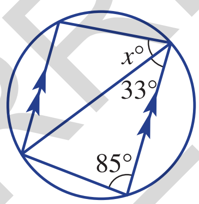

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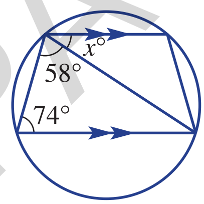

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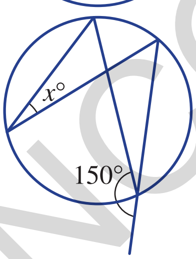

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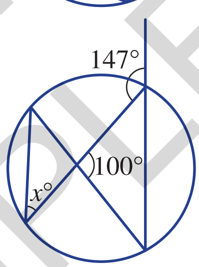
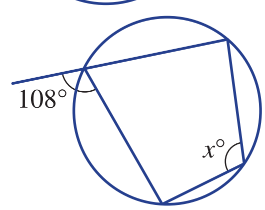
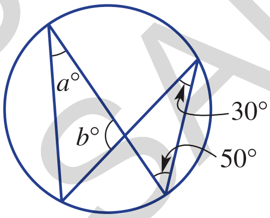


i



7 Find the values of the pronumerals in these circles.

a



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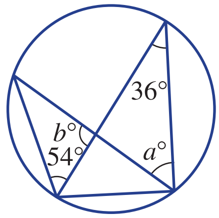
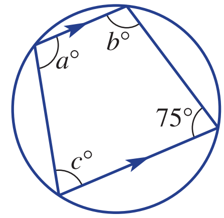


c



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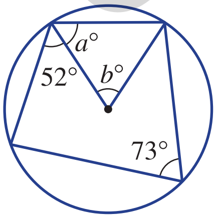

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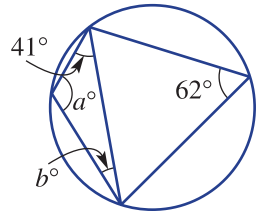

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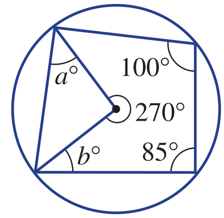

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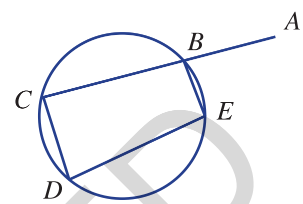
2H

	8	8, 9	9, 10

REASONING

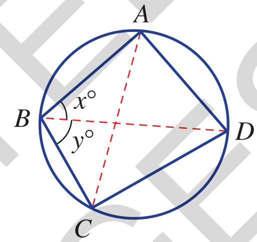
8 $\angle ABE$ is an exterior angle to the cyclic quadrilateral *BCDE*.

- **a** If $\angle ABE = 80°$, find $\angle CDE$.
- **b** If $\angle ABE = 71°$, find $\angle CDE$.
- **c** Prove that $\angle ABE = \angle CDE$ using circle theorem 4.



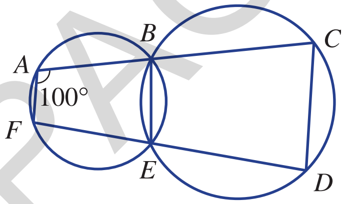
9 Prove that opposite angles in a cyclic quadrilateral are supplementary by following these steps.

- **a** Explain why $\angle ACD = x°$ and $\angle DAC = y°$.
- **b** Prove that $\angle ADC = 180° - (x + y)°$.
- **c** What does this say about $\angle ABC$ and $\angle ADC$?



10 If $\angle BAF = 100°$, complete the following.

- **a** Find:
  - **i** $\angle FEB$
  - **ii** $\angle BED$
  - **iii** $\angle DCB$
- **b** Explain why $AF \parallel CD$.

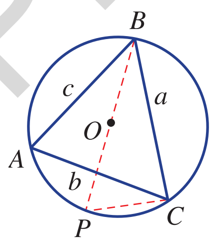


The sine rule

	—	—	11

ENRICHMENT

11 Consider a triangle *ABC* inscribed in a circle. The construction line *BP* is a diameter and *PC* is a chord. If *r* is the radius, then $BP = 2r$.

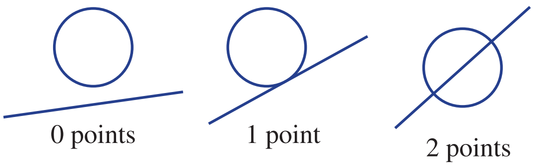


- **a** What can be said about $\angle PCB$? Give a reason.
- **b** What can be said about $\angle A$ and $\angle P$? Give a reason.
- **c** If $BP = 2r$, use trigonometry with $\angle P$ to write an equation linking *r* and *a*.
- **d** Prove that $2r = \dfrac{a}{\sin A}$, giving reasons.

2I Tangents EXTENDING

Interactive
Widgets
HOTsheets
Walkthroughs

When a line and a circle are drawn, three possibilities arise: they could intersect 0, 1 or 2 times.



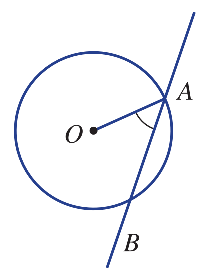
If the line intersects the circle once then it is called a tangent. If it intersects twice it is called a secant.


The surface of the road is a tangent to the tyre.

Let's start: From secant to tangent

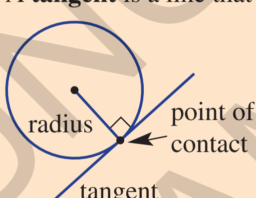
This activity is best completed using dynamic computer geometry software.

- Construct a circle with centre O and a secant line that intersects at A and B. Then measure $\angle BAO$.
- Drag B to alter $\angle BAO$. Can you place B so that line AB is a tangent? In this case what is $\angle BAO$?

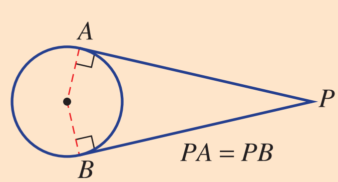


Key ideas

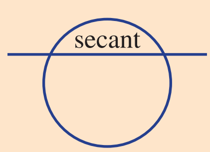
- A **tangent** is a line that touches a circle at a point called the **point of contact**.



  - A tangent intersects the circle exactly once.
  - A tangent is perpendicular to the radius at the point of contact.
  - Two different tangents drawn from an external point to the circle create line segments of equal length.



- A **secant** is a line that cuts a circle twice.

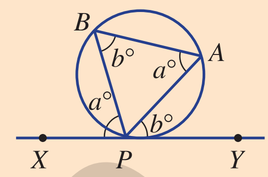


Key ideas

- **Alternate segment theorem**: The angle between a tangent and a chord is equal to the angle in the alternate segment.

  $\angle APY = \angle ABP$ and

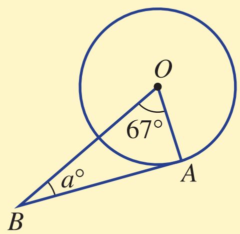
  $\angle BPX = \angle BAP$



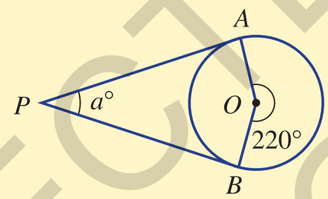
Example 15 Finding angles with tangents

Find the value of a in these diagrams that include tangents.

a



b



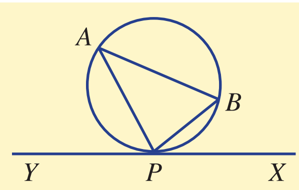
SOLUTION	EXPLANATION
a $\angle BAO = 90°$	BA is a tangent, so $OA \perp BA$.
$a + 90 + 67 = 180$	The sum of the angles in a triangle is 180°.
$\therefore\ a = 23$	
b $\angle PAO = \angle PBO = 90°$	$PA \perp OA$ and $PB \perp OB$.
Obtuse $\angle AOB = 360° - 220° = 140°$	Angles in a revolution sum to 360°.
$a + 90 + 90 + 140 = 360$	Angles in a quadrilateral sum to 360°.
$\therefore\ a = 40$	

Example 16 Using the alternate segment theorem

In this diagram XY is a tangent to the circle.

a Find $\angle BPX$ if $\angle BAP = 38°$.

b Find $\angle ABP$ if $\angle APY = 71°$.



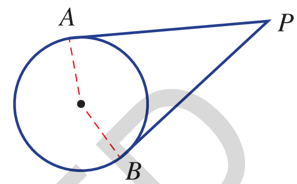
SOLUTION	EXPLANATION
a $\angle BPX = 38°$	The angle between a tangent and a chord is equal to the angle in the alternate segment.
b $\angle ABP = 71°$	

Exercise 2I

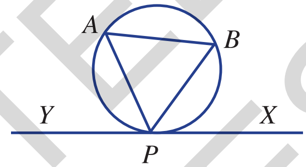
1–3	2, 3	—

UNDERSTANDING

1 **a** How many times does a tangent intersect a circle?
b At the point of contact, what angle does the tangent make with the radius?
c If AP is 5 cm, what is the length BP in this diagram?



2 For this diagram use the alternate segment theorem and name the angle that is:
a equal to $\angle BPX$ **b** equal to $\angle BAP$
c equal to $\angle APY$ **d** equal to $\angle ABP$



3 What is the interior angle sum for:
a a triangle? **b** a quadrilateral?

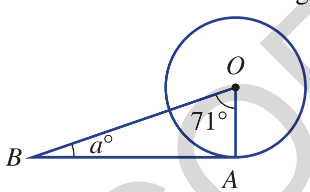
4, 5(½), 6, 7	4, 5(½), 6–8	4, 5(½), 6–8

FLUENCY

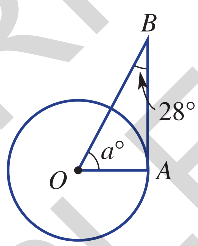
Example 15a

4 Find the value of a in these diagrams that include tangents.

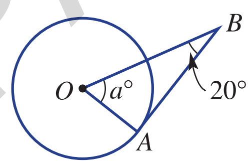
a



b



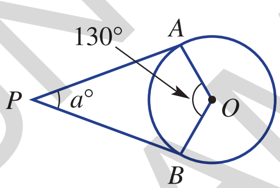
c



Example 15b

5 Find the value of a in these diagrams that include two tangents.

a



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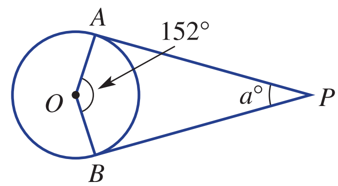

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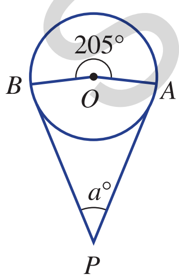

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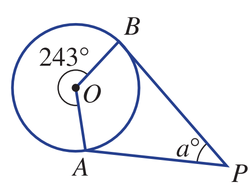


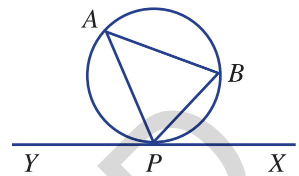
FLUENCY

Example 16

6 In this diagram, XY is a tangent to the circle. Use the alternate segment theorem to find:

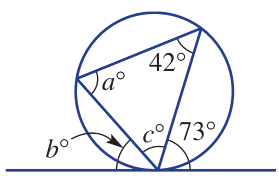
a $\angle PAB$ if $\angle BPX = 50°$

b $\angle APY$ if $\angle ABP = 59°$

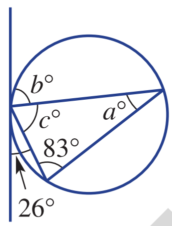


7 Find the value of a, b and c in these diagrams involving tangents.

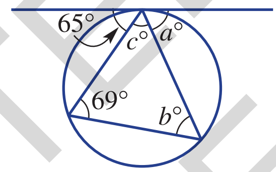
a



b

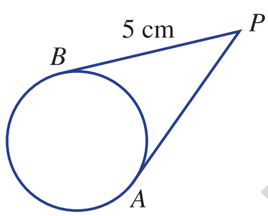


c



8 Find the length AP if AP and BP are both tangents.

a



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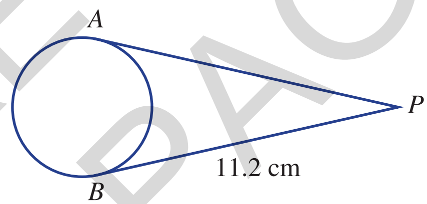
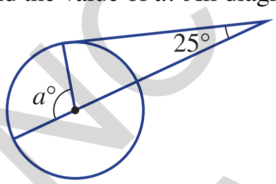


9–10(½)	9–10(½)	10(½), 11

PROBLEM-SOLVING

9 Find the value of a. All diagrams include one tangent line.

a



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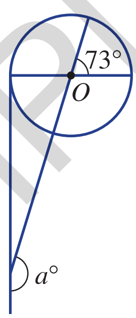

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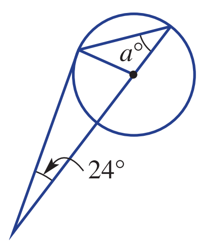

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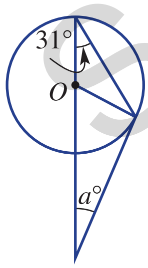

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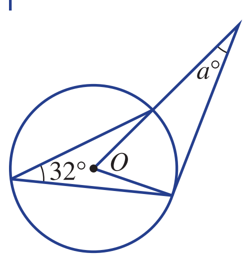

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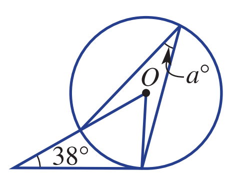

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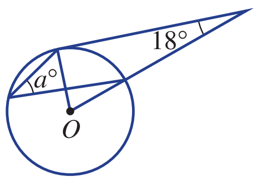

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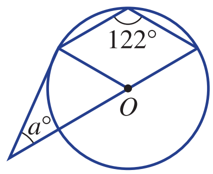
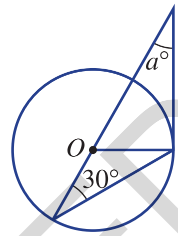
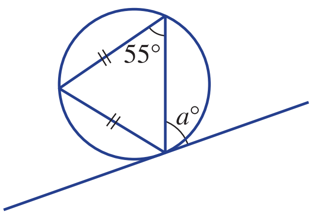


i

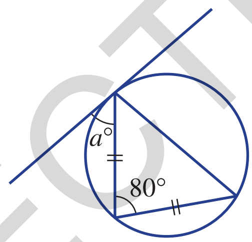


10 Find the value of a in these diagrams involving tangents.

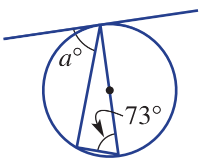
a



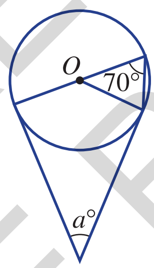
b



c



d



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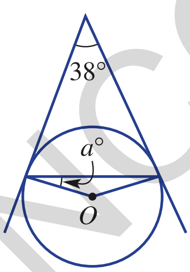
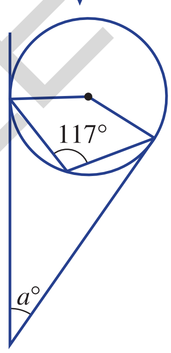
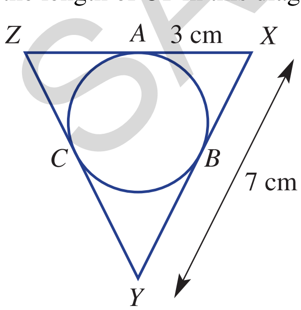


f



11 Find the length of CY in this diagram.



12, 13	12, 14	13–15

REASONING

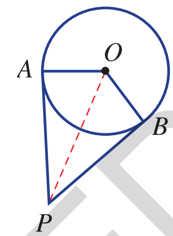
12 Prove that $AP = BP$ by following these steps.

a Explain why $OA = OB$.

b What is the size of $\angle OAP$ and $\angle OBP$?

c Hence, prove that $\triangle OAP \equiv \triangle OBP$.

d Explain why $AP = BP$.

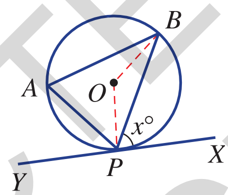


13 Prove the alternate angle theorem using these steps. First, let $\angle BPX = x^\circ$, then give reasons at each step.

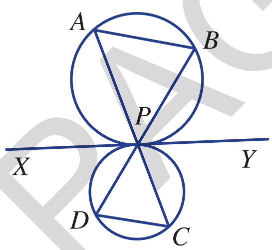
a Write $\angle OPB$ in terms of x.

b Write obtuse $\angle BOP$ in terms of x.

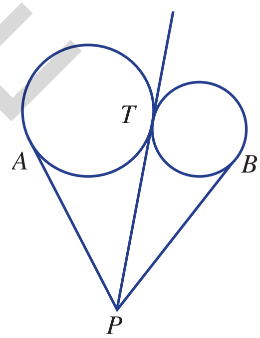
c Use circle theorem 1 from angle properties of a circle to write $\angle BAP$ in terms of x.



14 These two circles touch with a common tangent XY. Prove that $AB \parallel DC$. You may use the alternate segment theorem.



15 PT is a common tangent. Explain why $AP = BP$.



Bisecting tangent

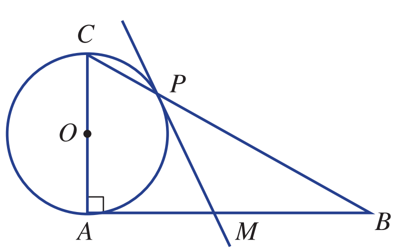
—	—	16

ENRICHMENT

16 In this diagram, $\triangle ABC$ is right angled, AC is a diameter and PM is a tangent at P, where P is the point at which the circle intersects the hypotenuse.

a Prove that PM bisects AB; i.e. that $AM = MB$.

b Construct this figure using dynamic geometry software and check the result. Drag A, B or C to check different cases.



2J Intersecting chords, secants and tangents EXTENDING

Interactive
Widgets
HOTsheets
Walkthroughs

In circle geometry, the lengths of the line segments (or intervals) formed by intersecting chords, secants or tangents are connected by special rules. There are three situations in which this occurs:

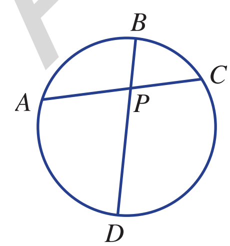
1 intersecting chords
2 intersecting secant and tangent
3 intersecting secants.

Architects use circle and chord geometry to calculate the dimensions of constructions, such as this glass structure.

Let's start: Equal products

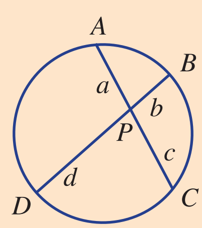
Use dynamic geometry software to construct this figure and then measure AP, BP, CP and DP.

- Calculate $AP \times CP$ and $BP \times DP$. What do you notice?
- Drag A, B, C or D. What can be said about $AP \times CP$ and $BP \times DP$ for any pair of intersecting chords?

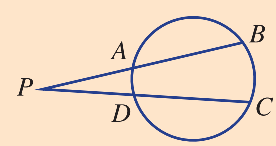


Key ideas

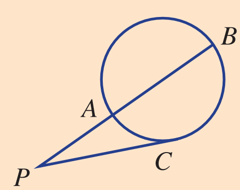
- When two chords intersect as shown, then $AP \times CP = BP \times DP$ or $ac = bd$.



- When two secants intersect at an external point P as shown, then $AP \times BP = DP \times CP$.



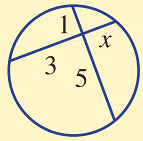
- When a secant intersects a tangent at an external point as shown, then $AP \times BP = CP^2$.



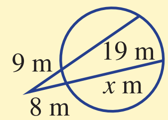
Example 17 Finding lengths using intersecting chords, secants and tangents

Find the value of x in each figure.

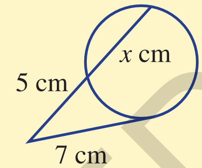
a



b



c



SOLUTION	EXPLANATION
a $x \times 3 = 1 \times 5$ $3x = 5$ $x = \frac{5}{3}$	Equate the products of each pair of line segments on each chord.
b $8 \times (x + 8) = 9 \times 28$ $8x + 64 = 252$ $8x = 188$ $x = \frac{188}{8}$ $= \frac{47}{2}$	Multiply the entire length of the secant by the length from the external point to the first intersection point with the circle. Then equate both products. Expand brackets and solve for x.
c $5 \times (x + 5) = 7^2$ $5x + 25 = 49$ $5x = 24$ $x = \frac{24}{5}$	Square the length of the tangent and then equate with the product from the other secant.

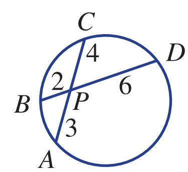
Exercise 2J

1–3	3	—

UNDERSTANDING

1 State these lengths for the given diagram.

a AP **b** DP **c** AC **d** BD



2 Solve these equations for x.

a $x \times 2 = 7 \times 3$

b $4x = 5 \times 2$

c $(x + 3) \times 7 = 6 \times 9$

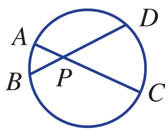
d $7(x + 4) = 5 \times 11$

2J

UNDERSTANDING

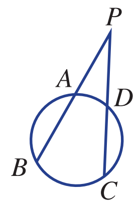
3 Complete the rules for each diagram.

a



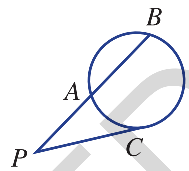
$AP \times CP =$ _____ $\times$ _____

b



$AP \times$ _____ $=$ _____ $\times$ _____

c



$AP \times$ _____ $=$ _____^{2}

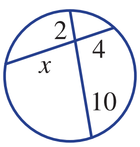
4–6(½)	4–6(½)	4–7(½)

FLUENCY

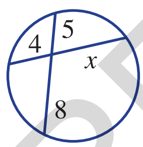
Example 17a

4 Find the value of x in each figure.

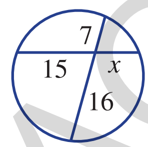
a



b



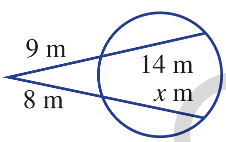
c



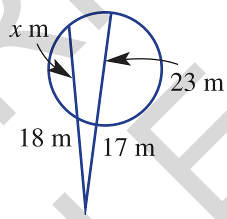
Example 17b

5 Find the value of x in each figure.

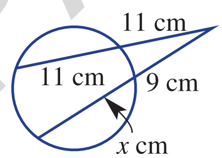
a



b



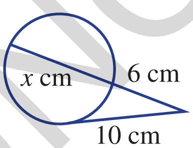
c



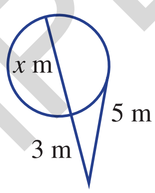
Example 17c

6 Find the value of x in each figure.

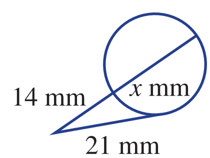
a



b

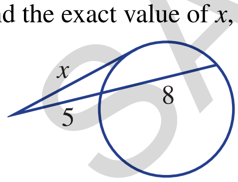


c

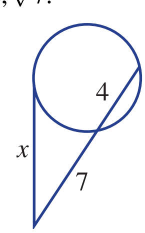


7 Find the exact value of x, in surd form. For example, $\sqrt{7}$.

a



b

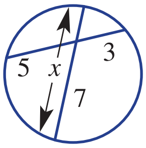


8(½)	8(½)	8(½), 9

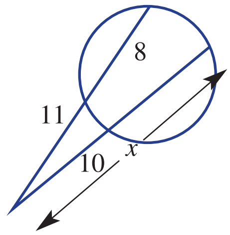
PROBLEM-SOLVING

8 Find the exact value of x.

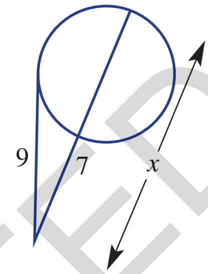
a



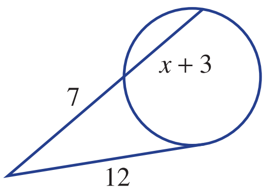
b



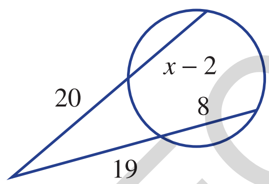
c



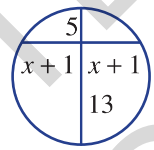
d



e

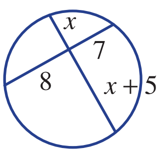


f

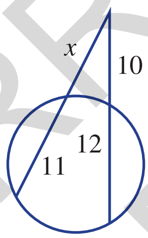


9 For each diagram, derive the given equations.

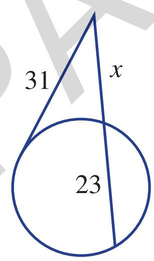
a $x^2 + 5x - 56 = 0$



b $x^2 + 11x - 220 = 0$



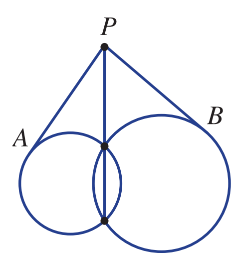
c $x^2 + 23x - 961 = 0$



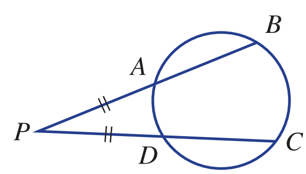
10, 11	10–12	12–14

REASONING

10 Explain why $AP = BP$ in this diagram, using your knowledge from this section.



11 In this diagram $AP = DP$. Explain why $AB = DC$.



2J

REASONING

12 Prove that $AP \times CP = BP \times DP$ by following these steps.

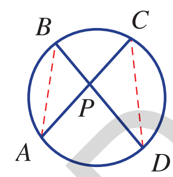
a What can be said about the pair of angles $\angle A$ and $\angle D$ and also about the pair of angles $\angle B$ and $\angle C$? Give a reason.

b Prove $\triangle ABP \;|||\; \triangle DCP$.

c Complete:

$$\frac{AP}{....} = \frac{....}{CP}$$

d Prove $AP \times CP = BP \times DP$.

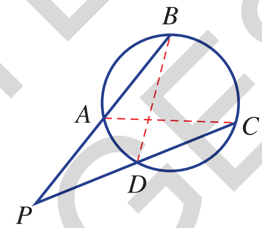


13 Prove that $AP \times BP = DP \times CP$ by following these steps.

a Consider $\triangle PBD$ and $\triangle PCA$. What can be said about $\angle B$ and $\angle C$? Give a reason.

b Prove $\triangle PBD \;|||\; \triangle PCA$.

c Prove $AP \times BP = DP \times CP$.



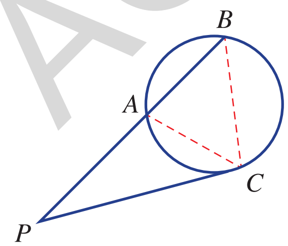
14 Prove that $AP \times BP = CP^2$ by following these steps.

a Consider $\triangle BPC$ and $\triangle CPA$. Is $\angle P$ common to both triangles?

b Explain why $\angle ACP = \angle ABC$.

c Prove $\triangle BPC \;|||\; \triangle CPA$.

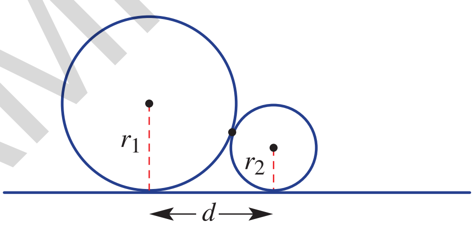
d Prove $AP \times BP = CP^2$.



Horizontal wheel distance	—	—	15

ENRICHMENT

15 Two touching circles have radii r_1 and r_2. The horizontal distance between their centres is d. Find a rule for d in terms of r_1 and r_2.



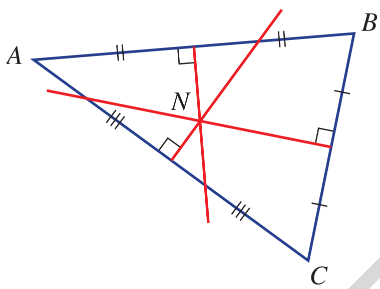
Investigation

Some special points of triangles and Euler's line

From Year 9, you may recall finding the circumcentre and the centroid of a triangle. Here we will review the construction of these two points and consider in further detail some of their properties, as well as their relationship with a third point – the orthocentre. This is best done using a dynamic computer geometry software package.

The circumcentre of a triangle

The perpendicular bisectors of each of the three sides of a triangle meet at a common point called the **circumcentre**.



a Construct and label a triangle ABC and measure each of its angles.

b Construct the perpendicular bisector of each side of the triangle.

c Label the point of intersection of the lines N. This is the circumcentre.

d By dragging the points of your triangle, observe what happens to the location of the circumcentre. Can you draw any conclusions about the location of the circumcentre for some of the different types of triangles; for example, equilateral, isosceles, right-angled or obtuse?

e Construct a circle centred at N with radius NA. This is the **circumcircle**. Drag the vertex A to different locations. What do you notice about vertices B and C in relation to this circle?

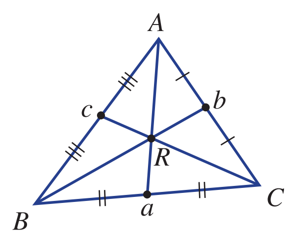
The centroid of a triangle

The three medians of a triangle intersect at a common point called the **centroid**. A **median** is the line drawn from a vertex to the midpoint of the opposite side.

a Construct and label a new triangle ABC and measure each of its angles.

b Mark the midpoint of each side of the triangle and construct a segment from each vertex to the midpoint of the opposite side.

c Observe the common point of intersection of these three medians and label this point R – the centroid. Point R is the centre of gravity of the triangle.



d Label the points of your triangle as shown and complete the table below by measuring the required lengths. Drag the vertices of your triangle to obtain different triangles to use for the table.

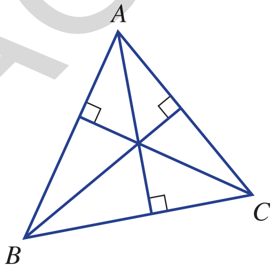
Lengths	*AR*	*Ra*	*BR*	*Rb*	*CR*	*Rc*
Triangle 1						
Triangle 2						
Triangle 3						
Triangle 4						

e From your table, what can you observe about how far along each median the centroid lies?

f Explain why each median divides the area of triangle *ABC* in half. What can be said about the area of the six smaller triangles formed by the three medians?

The orthocentre of a triangle

The three altitudes of a triangle intersect at a common point called the **orthocentre**. An **altitude** of a triangle is a line drawn from a vertex to the opposite side of the triangle, meeting it at right angles.



a Construct a triangle *ABC* and measure each angle.

b For each side, construct a line that is perpendicular to it and that passes through its opposite vertex.

c Label the point where these three lines intersect as *O*. This is the orthocentre.

d By dragging the vertices of the triangle, comment on what happens to the location of the orthocentre for different types of triangles.

e Can you create a triangle for which the orthocentre is outside the triangle? Under what circumstances does this occur?

All in one

a Construct a large triangle *ABC* and measure the angles. On this one triangle use the previous instructions to locate the circumcentre (*N*), the centroid (*R*) and the orthocentre (*O*).

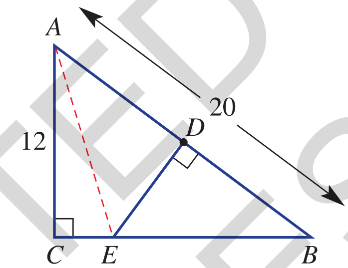
b By dragging the vertices, can you make these three points coincide? What type of triangle achieves this?

c Construct a line joining the points *N* and *R*. Drag the vertices of the triangle. What do you notice about the point *O* in relation to this line?

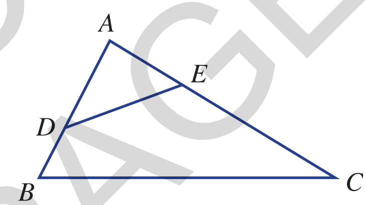
d The line in part **c** is called **Euler's line**. Use this line to determine the ratio of the length *NR* to the length *RO*.

Problems and challenges

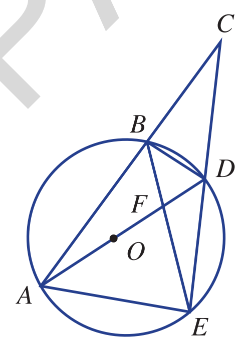
1 In a triangle ABC, angle C is a right angle, D is the midpoint of AB and DE is perpendicular to AB. The length of AB is 20 and the length of AC is 12. What is the area of triangle ACE?



2 In this diagram, $AB = 15$ cm, $AC = 25$ cm, $BC = 30$ cm and $\angle AED = \angle ABC$. If the perimeter of $\triangle ADE$ is 28 cm, find the lengths of BD and CE.



3 Other than straight angles, name all the pairs of equal angles in the diagram shown.



4 A person stands in front of a cylindrical water tank and has a viewing angle of 27° to the sides of the tank. What percentage of the circumference of the tank can they see?

5 An isosceles triangle ABC is such that its vertices lie on the circumference of a circle. $AB = AC$ and the chord from A to the point D on the circle intersects BC at E. Prove that $AB^2 - AE^2 = BE \times CE$.

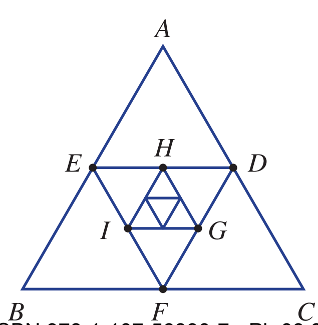
6 D, E and F are the midpoints of the three sides of $\triangle ABC$. The straight line formed by joining two midpoints is parallel to the third side and half its length.

a Prove $\triangle ABC$ ||| $\triangle FDE$.
$\triangle GHI$ is drawn in the same way such that G, H and I are the midpoints of the sides of $\triangle DEF$.

b Find the ratio of the area of:

- **i** $\triangle ABC$ to $\triangle FDE$
- **ii** $\triangle ABC$ to $\triangle HGI$

c Hence, if $\triangle ABC$ is the first triangle drawn, what is the ratio of the area of $\triangle ABC$ to the area of the nth triangle drawn in this way?



Chapter summary

Geometry

Similar figures

All corresponding angles are equal, corresponding sides are in the same ratio; i.e. same shape but different in size.
Tests for similar triangles:
SSS, SAS, AAA, RHS.
Written as $\triangle ABC \,|||\, \triangle DEF$ or
$\triangle ABC \sim \triangle DEF$.

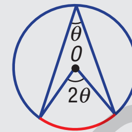
Polygons

Angle sum of polygon
$S = 180(n - 2)$, where n is the number of sides.

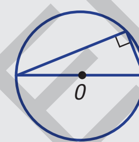
Congruent triangles

These triangles are identical, written $\triangle ABC \equiv \triangle DEF$.
Tests for congruence are
SSS, SAS, AAS and RHS.

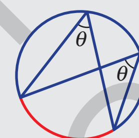
Angle properties of circles 10A



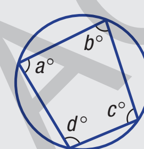
Angle at centre is twice angle at the circumference subtended by the same arc.



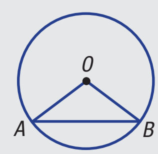
The angle in a semicircle is 90°.



Angles at circumference subtended by the same arc are equal.

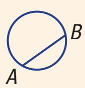


Opposite angles in cyclic quadrilaterals are supplementary.
$a + c = 180$
$b + d = 180$



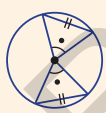
$\triangle OAB$ is isosceles given OA and OB are radii.

Circles and chords 10A

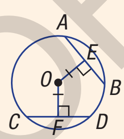


AB is a chord.

Circle theorem 1
Chords of equal length subtend equal angles.

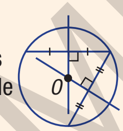


Circle theorem 2
If $AB = CD$,
then $OE = OF$.

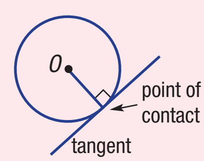


Circle theorem 3
Perpendicular from centre to chord bisects chord and angle at O.

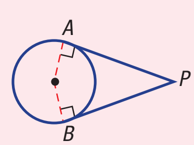
Circle theorem 4
Perpendicular bisectors of every chord of a circle intersect at the centre.



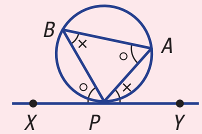
Tangents Ext



A tangent touches a circle once and is perpendicular to the radius at point of contact.

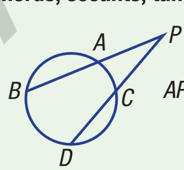


Tangents PA and PB have equal length; i.e. $PA = PB$.

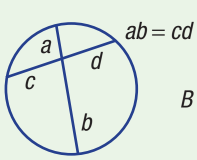


Alternate segment theorem: angle between tangent and chord is equal to the angle in the alternate segment.

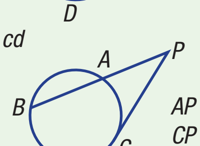
Intersecting chords, secants, tangents Ext



$AP \times BP = CP \times DP$



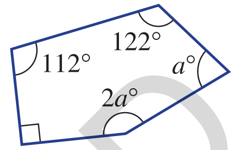
$ab = cd$



$AP \times BP = CP^2$
CP is a tangent.

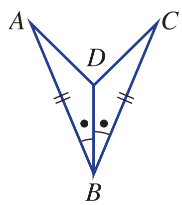
Multiple-choice questions

2A **1** The value of a in the polygon shown is:



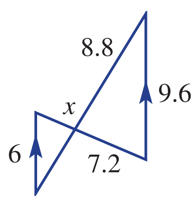
A 46 **B** 64 **C** 72
D 85 **E** 102

2B **2** The test that proves that $\triangle ABD \equiv \triangle CBD$ is:



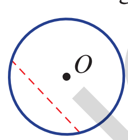
A RHS **B** SAS **C** SSS **D** AAA **E** AAS

2D **3** The value of x in the diagram shown is:



A 4.32 **B** 4.5 **C** 3.6 **D** 5.5 **E** 5.2

2F 10A **4** The name given to the dashed line in the circle with centre O is:

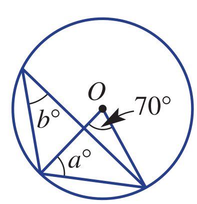


A a diameter **B** a minor arc **C** a chord
D a tangent **E** a secant

2F 10A **5** A circle of radius 5 cm has a chord 4 cm from the centre of the circle. The length of the chord is:

A 4.5 cm **B** 6 cm **C** 3 cm
D 8 cm **E** 7.2 cm

2G 10A **6** The values of the pronumerals in the diagram are:



A $a = 55, b = 35$
B $a = 30, b = 70$
C $a = 70, b = 35$
D $a = 55, b = 70$
E $a = 40, b = 55$

2H 10A

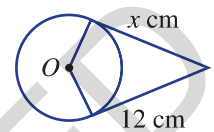
7 A cyclic quadrilateral has one angle measuring 63° and another angle measuring 108°. Another angle in the cyclic quadrilateral is:

A 63° **B** 108° **C** 122°
D 75° **E** 117°

2I Ext

8 For the circle shown at right with radius 5 cm, the value of x is:

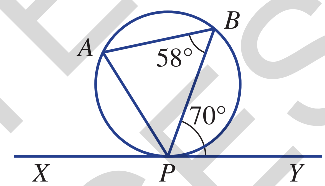
A 13 **B** 10.9 **C** 12
D 17 **E** 15.6



2I Ext

9 By making use of the alternate segment theorem, the value of $\angle APB$ is:

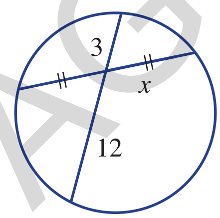
A 50° **B** 45° **C** 10°
D 52° **E** 25°



2J Ext

10 The value of x in the diagram is:

A 7.5 **B** 6 **C** 3.8
D 4 **E** 5

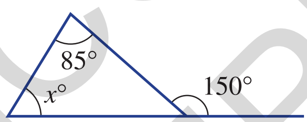


Short-answer questions

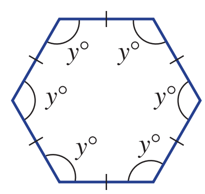
2A

1 Determine the value of each pronumeral.

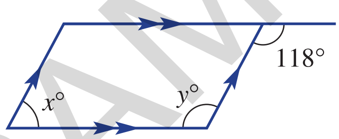
a



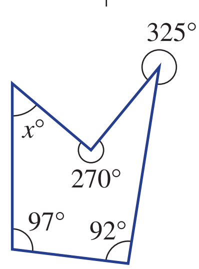
b



c



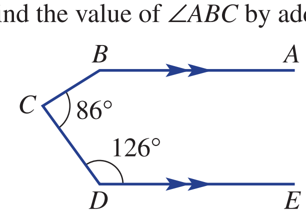
d



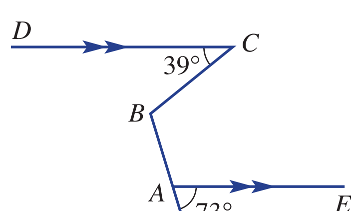
2A

2 Find the value of $\angle ABC$ by adding a third parallel line.

a

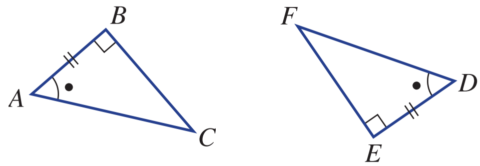


b

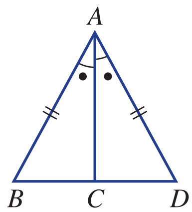


2B **3** Prove that each pair of triangles is congruent, giving reasons.

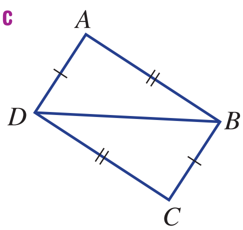
a



b



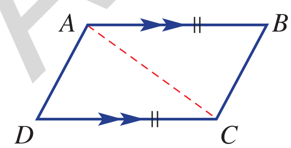
c



2C **4** Complete these steps to prove that if one pair of opposite sides is equal and parallel in a quadrilateral, then it is a parallelogram.

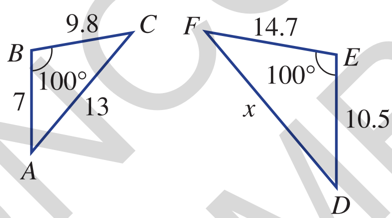
a Prove $\triangle ABC \equiv \triangle CDA$, giving reasons.

b Hence, prove $AD \parallel BC$.

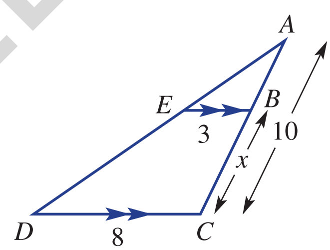


2E **5** In each of the following, identify pairs of similar triangles by proving similarity, giving reasons, and then use this to find the value of x.

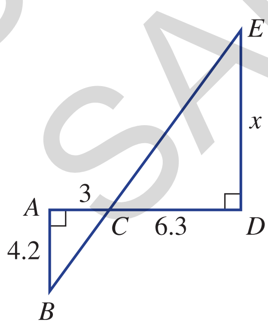
a



b



c



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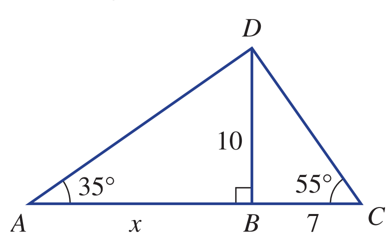
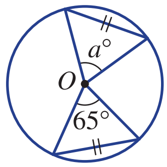


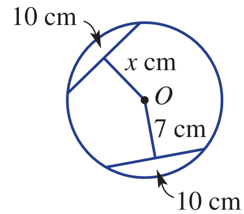
2F 10A

6 Find the value of each pronumeral and state the chord theorem used.

a



b



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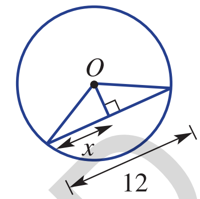
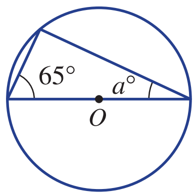


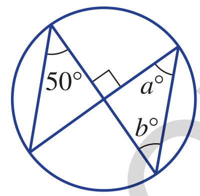
2G/H 10A

7 Use the circle theorems to help find the values of the pronumerals.

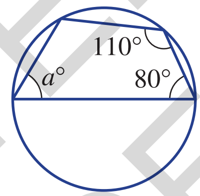
a



b



c



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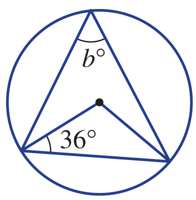
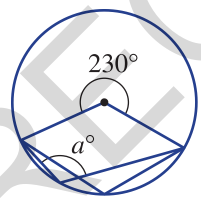
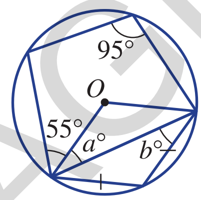


e



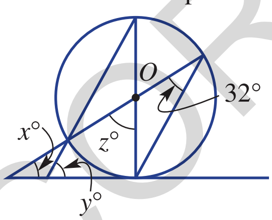
f



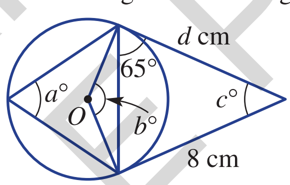
2I Ext

8 Find the value of the pronumerals in these diagrams involving tangents and circles.

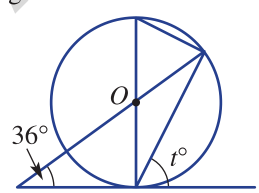
a



b



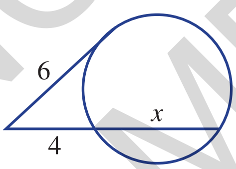
c



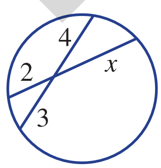
2J Ext

9 Find the value of x in each figure.

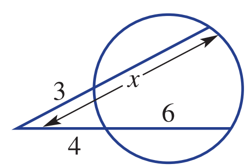
a



b



c



Extended-response questions

1 The triangular area of land shown is to be divided into two areas such that $AC \parallel DE$. The land is to be divided so that $AC : DE = 3 : 2$.

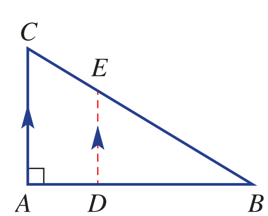
a Prove that $\triangle ABC \ ||| \ \triangle DBE$.

b If $AC = 1.8$ km, find DE.

c If $AD = 1$ km and $DB = x$ km:

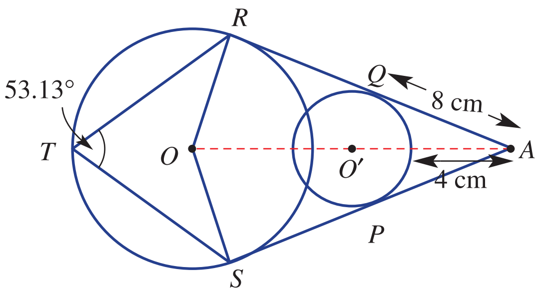
  i Show that $2(x + 1) = 3x$. **ii** Solve for x.

d For the given ratio, what percentage of the land area does $\triangle DBE$ occupy? Answer to one decimal place.



10A

2 The diagram below shows two intersecting circles sharing common tangents AR and AS. The distance between the centres O and O' of the two circles is 15 cm. Other measurements are as shown. Given that the two centres O and O' and the point A are in a straight line, complete the following.



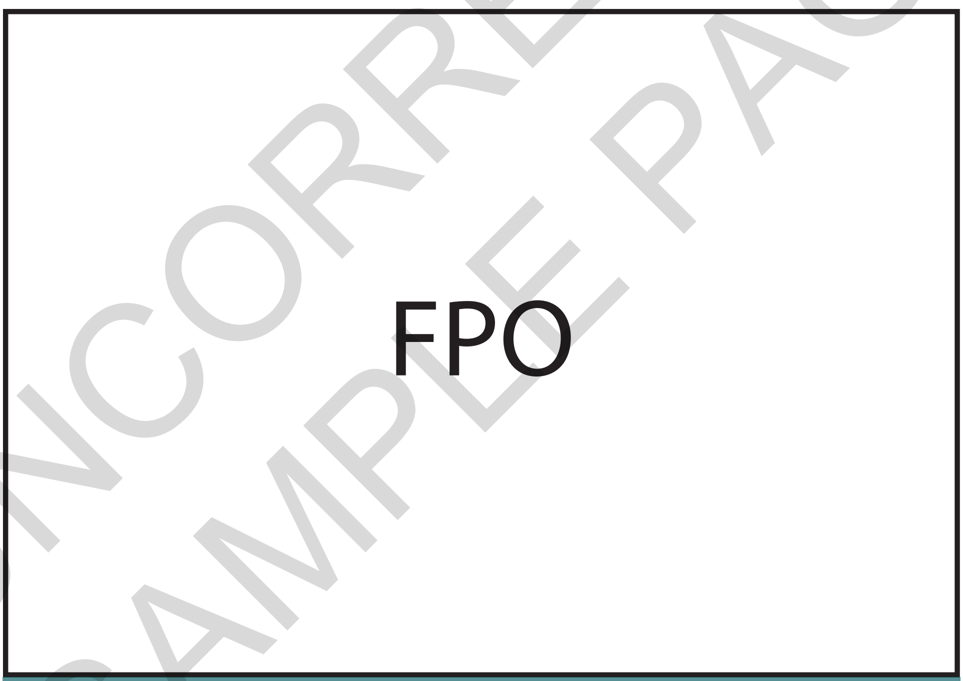
a Find the values of these angles.

  i $\angle ROS$ **ii** $\angle RAS$

b Use the rule for intersecting secants and tangents to help find the diameter of the smaller circle.

c Hence, what is the distance from A to O?

d By first finding AR, determine the perimeter of $AROS$. Round to one decimal place.



Two water ripples intersecting.