MATHEMATICAL PROOF AND REASONING

4

Syllabus Checklist

By the end of this chapter, you should be able to:

- ☐ prove simple results involving numbers (ACMSM061)
- ☐ express rational numbers as terminating or eventually recurring decimals and vice versa (ACMSM062)
- ☐ prove irrationality by contradiction for numbers such as $\sqrt{2}$ and $\log_2 5$ (ACMSM063)
- ☐ develop the nature of inductive proof including the 'initial statement' and inductive step (ACMSM064)
- ☐ prove results for sums, such as $1 + 4 + 9 ... + n^2 = \frac{n(n+1)(2n+1)}{6}$ for any positive integer n (ACMSM065)
- ☐ prove divisibility results, such as $3^{2n+4} - 3^{2n}$ is divisible by 5 for any positive integer n. (ACMSM066)

4.1 INTRODUCTION TO NUMBER PATTERNS AND DEFINITIONS

You will be familiar with the following sets of numbers:

Natural Numbers $N = \{1,2,3,4,5,6...\}$

Whole Numbers $W = \{0,1,2,3,4,5,...\}$

Integers $I = \{...,-4,-3,-2,-1,0,1,2,3,4,...\}$

Prime Numbers $P = \{2,3,5,7,11,13,...\}$

Note here that the dots in a set (i.e. ...) normally mean "and so on continuing with the same pattern", but in the case of the set P it means continue on listing the prime numbers as there is no known pattern.

A ***conjecture*** is the expression of an opinion without sufficient evidence. Conjectures may be true or false. Only one counter example is necessary to prove that a conjecture is false.

An ***axiom*** is a statement that is accepted as being true without proof.

A ***theorem*** is a statement that has been proven to be true.

A ***proposition*** is a statement that can only be either true or false.

A ***definition*** is a statement of what we agree about something. For example, one definition of an even number is:

An ***even number*** is a natural number which has 2 as a factor,

$$\therefore\ E = \{2,4,6,8,...\} \text{ is the set of even numbers.}$$

An ***odd number*** could be defined as a natural number which does not have 2 as a factor,

$$\therefore\ O = \{1,3,5,7,9,...\} \text{ is the set of odd numbers.}$$

(For the purposes of this text the author has chosen not to include zero in the set of even numbers and also not to include the negative integers in the sets E and O.)

If n is a natural number (i.e. $n \in N$), then $2n$ is always even and $2n + 1$ or $2n - 1$ is always odd. The sequence of consecutive natural numbers starting with n is

$$n,\ n + 1,\ n + 2,\ n + 3, \ldots$$

The sequence of consecutive even numbers starting with $2n$ for any $n \in N$ is

$$2n,\ 2(n + 1),\ 2(n + 2),\ 2(n + 3), \ldots$$

The sequence of consecutive odd numbers starting with $2n - 1$ for any $n \in N$ is

$$2n - 1, 2n + 1, 2n + 3, 2n + 5, \ldots$$

A ***terminating decimal*** is a decimal number that has digits that do not go on forever (e.g. 0.25 has two digits before terminating; 2.0451 has four digits before terminating).

A ***recurring decimal*** is a decimal number that has digits that recur forever in a recognisable pattern (e.g. $\frac{1}{3} = 0.\dot{3}$ where the 3s recur forever; $\frac{1}{11} = 0.\overline{09}$ where the '09' digits will recur forever).

4.2 CONVERTING RECURRING DECIMALS TO FRACTIONS

Worked Example 1

Convert the following recurring decimals to fractions

(a) $0.\dot{4}$ (b) $0.\overline{08}$ (c) $0.58\dot{3}$

Worked Solutions

(a) Step 1 Let x = $0.\dot{4}$

Step 2 Multiply both sides of the equation in Step 1 by 10 (this is because there is one zero in 10 and one repeating digit)

$$10x = 4.\dot{4}$$

Step 3 Subtract x from both sides of the equation and simplify

$$10x - 1x = 4.\dot{4} - 0.\dot{4}$$

$$9x = 4$$

$$x = \frac{4}{9}$$

(b) Step 1 Let x = $0.\overline{08}$

Step 2 Multiply both sides of the equation in Step 1 by 100 (this is because there are two zeroes in 100 and two repeating digits)

$$100x = 8.\overline{08}$$

Step 3 Subtract x from both sides of the equation and simplify

$$100x - 1x = 8.\overline{08} - 0.\overline{08}$$

$$99x = 8$$

$$x = \frac{8}{99}$$

(c) Step 1 Let $x = 0.58\dot{3}$

Step 2 Multiply both sides of the equation in Step 1 by 10 (this is because there is one zero in 10 and one repeating digit)

$$10x = 5.8\dot{3}$$

Step 3 Subtract x from both sides of the equation and simplify

$$10x - 1x = 5.8\dot{3} - 0.58\dot{3}$$

$$9x = 5.25$$

$$x = \frac{5.25}{9}$$

Step 4 Multiply answer by $\frac{100}{100}$ to remove two decimal places, and simplify

$$x = \frac{525}{900}$$

$$\therefore x = \frac{7}{12}$$

4.3 CONVERTING RECURRING DECIMALS TO FRACTIONS

Worked Example 2

Convert the following fractions to decimals and indicate appropriately any recurring digits

(a) $\frac{8}{9}$ (b) $\frac{9}{22}$

Worked Solutions

(a) Step 1 Prepare the fraction for short division

$$9\overline{)8}$$

Step 2 Begin by dividing 9 into 8 (which can't be done evenly). Place a zero above the 8, followed by a decimal point. Also place a decimal point after the 8 followed by a decimal point.

$$\begin{array}{r} 0. \\ 9\overline{)8.0} \end{array}$$

Step 3 Now divide 9 into 80 (which has been created by the previous operations). Place the dividend above the zero, and write the remainder in front of another zero placed after the '80'.

$$\begin{array}{r} 0.88 \\ 9\overline{)8.0{}^{8}0} \end{array}$$

Step 4 Continue this process until the pattern of recurring decimals appears

$$\begin{array}{r}0.888\\9\overline{\smash{)}8.0^{8}0^{8}0}\end{array}$$

Step 5 Conclude by writing the decimal correctly $0.\dot{8}$

(b) Step 1 Prepare the fraction for short division $22\overline{\smash{)}9}$

Step 2 Begin by dividing 22 into 9 (which can't be done evenly). Place a zero above the 9, followed by a decimal point. Also place a decimal point after the 9 followed by a decimal point.

$$\begin{array}{r}0.\\22\overline{\smash{)}9.0}\end{array}$$

Step 3 Now divide 22 into 90 (which has been created by the previous operations). Place the dividend above the zero, and write the remainder in front of another zero placed after the '90'.

$$\begin{array}{r}0.40\\22\overline{\smash{)}9.0^{2}0}\end{array}$$

Step 4 Continue this process until the pattern of recurring decimals appears

$$\begin{array}{r}0.409\\22\overline{\smash{)}9.0^{2}0\ 0}\end{array}$$

Step 5 Conclude by writing the decimal correctly $0.4\overline{09}$

4.4 PROOFS USING ALGEBRA

Algebra can be used to prove if a conjecture is true or false.

Worked Example 3

Conjecture: The sum of five consecutive odd numbers is a multiple of five.

Let $2n+1$, $2n + 3$, $2n + 5$, $2n + 7$, $2n + 9$ represent five consecutive odd numbers.

$$\begin{aligned}2n+1+ 2n + 3 + 2n + 5 + 2n + 7 + 2n + 9 &= 10n + 25\\&= 5(2n + 5)\end{aligned}$$

$\therefore 5(2n + 5)$ is divisible by 5 and hence is a multiple of 5. Therefore the conjecture is true.

4.5 PROOF BY CONTRADICTION

Contradiction is a form of proof that establishes the truth or validity of a proposition. It achieves this by showing that the proposition's being false would imply a contradiction.

Worked Example 4

Prove that $\sqrt{2}$ is irrational.

Step 1 Assume that $\sqrt{2}$ is rational i.e. $\sqrt{2} = \frac{a}{b}$ where the natural numbers a and b have no common factors besides 1 ($\frac{a}{b}$ has been cancelled down to the lowest terms). The aim of the proof is to contradict this assumption.

Step 2 If $\sqrt{2} = \frac{a}{b}$ then $2 = \frac{a^2}{b^2}$ and $a^2 = 2b^2$ which means that a^2 is a multiple of 2.

Step 3 If a^2 is a multiple of 2, then it needs to be shown that this means that a is also a multiple of 2.

i.e. Let $a = 2n + 1$

So $a^2 = 4n^2 + 4n + 1$

$= 2(2n^2 + 2n) + 1$

$= 2m + 1$ where $m = 2n^2 + 2n$

which means that a^2 is not a multiple of 2. The reasoning then proceeds:

If a is not a multiple of 2 then a^2 is not a multiple of 2 is true – which means that

If a^2 is a multiple of 2 then a is a multiple of 2 is true, by the *contrapositive* method of reasoning outlined above.

Step 4 If a is a multiple of 2, then $a = 2d$ for some natural number d,

i.e. $a^2 = 4d^2$ but $a^2 = 2b^2$

$\therefore 2b^2 = 4d^2$

$b^2 = 2d^2$

This means that b^2 is a multiple of 2, and by the same line of reasoning above, b must also be a multiple of 2.

Step 5 It has been shown that both a and b are multiples of 2 as they have the common factor of 2. This contradicts the original assumption that $\frac{a}{b}$ have no common factors. So $\sqrt{2}$ cannot be written as a fraction and hence $\sqrt{2}$ is irrational.

4.6 PROOF BY INDUCTION

A significant amount of mathematics involves the examination of patterns. Many of these patterns are concerned with generalisations about sequences and series.

Mathematical induction is a method of proof that is based in recursion, and it is used for proving conjectures which claim that a certain statement is true for integer values of some variable.

Let's say we were interested in finding a generalisation to explain the sum of n odd numbers, starting at 1. If we tabulate our findings for the first 10 odd numbers and their partial sums, we have:

n	1	2	3	4	5	6	7	8	9	10
T_n	1	3	5	7	9	11	13	15	17	19
S_n	1	4	9	16	25	36	49	64	81	100
n^2	1^2	2^2	3^2	4^2	5^2	6^2	7^2	8^2	9^2	10^2

The interesting thing here is that the last row of the table shows all integers n^2, $n \geq 1$. Thus, the sum of all n odd numbers appears to be the square of n. In stating this, we have arrived at a conjecture – which is the first step in creating a theorem – but we may not know precisely why this is true.

The following worked exercise provides a precise mathematical statement of the result we are trying to prove.

Worked Example 5 (General Series)

Prove by mathematical induction that for all odd integers $n \geq 1$ that

$1 + 3 + 5 + 7 + \dots + (2n - 1) = n^2$

Worked Solution

1. Initial step: We need to show that the hypothesis is true for a small value of n, e.g. $n = 1$. Substitute this value into the series.

$$1 = 1^2$$

2. Base step: We let $n = k$, which will assume that the statement will hold true for all values of n. After substituting $n = k$ into the series, we assume that the hypothesis is true for $n = k$.

$$1 + 3 + 5 + \dots (2k - 1) = k^2 \qquad \rightarrow (1)$$

Assume that base step is true, i.e. statement is true for $n = k$.

3. Base induction step: We let $n = k + 1$, to show that the statement will remain true for all values of k and the very next value after k. Note that this is not simply a case of substituting $k + 1$ for n in the original statement; doing so would be making another assumption. Instead, we add the next term of the series to both sides of the statement created in the 'base step'. This next term is $2(k + 1) -1$.

Looking back at (1), we can see that the series $1 + 3 + 5 \dots (2k - 1)$ exists in (2).

We thus substitute k^2 for $1 + 3 + 5 + \dots (2k - 1)$

$$1 + 3 + 5 + \dots (2k - 1) + [2(k + 1) -1] = (k+1)^2$$

$$\begin{aligned}
1 + 3 + 5 + \dots (2k - 1) + (2k + 1) &= (k+1)^2 \qquad \rightarrow (2)\\
\text{So } \underline{1 + 3 + 5 + \dots (2k - 1)} + (2k + 1) &= (k+1)^2\\
k^2 + (2k + 1) &= (k+1)^2\\
k^2 + 2k + 1 &= k^2 + 2k + 1\\
&= \text{R.H.S.}
\end{aligned}$$

4. Conclusion: Because we have proven that the LHS of the statement equals the RHS, we can conclude that the statement is true for all values of $n = k + 1$. Similarly, because the statement is true for all values of $n = k + 1$, it must also be true for all values of $n = k$. Therefore, the conjecture has been proven.

$\therefore$ true for $n = k + 1$

$\therefore$ true for $n = k$

$\therefore$ conjecture is true.

Worked Example 6 (Divisibility)

Using mathematical induction, prove that $3^{2n} -1$ is divisible by 8.

Worked Solution

1. Initial step: We need to show that the hypothesis is true for a small value of n, e.g. $n = 1$. Substitute this value into the expression, and equate the expression to some multiple of 8.

Let $3^{2n} -1 = 8A$ for some integer A

$3^{2(1)} -1 = 3^2 -1= 9 -1 = 8$

$\therefore$ this is a multiple of 8

$\therefore$initial step is true for $n = 1$

2. Base step: We let $n = k$, which will assume that the statement will hold true for all values of n. After substituting $n = k$ into the statement, we assume that the hypothesis is true for $n = k$.

$$3^{2k} - 1 = 8A$$

$$3^{2k} = 8A + 1 \quad \rightarrow (1)$$

Assume that base step is true, i.e. statement is true for $n = k$.

3. Base induction step: We let $n = k + 1$, to show that the statement will remain true for all values of k and the very next value after k. Here we substitute $n = k + 1$ into the statement, but we do not equate it to a multiple of 8. Instead, we will manipulate the expression so that something from (1) can be used to help prove the conjecture.

$$\begin{aligned} 3^{2(k+1)} - 1 &= 3^{2k+2} - 1 \\ &= 3^{2k}.\ 3^2 - 1 \\ &= 9(3^{2k}) - 1 \\ &= 9(8A + 1) - 1 \quad [\text{sub. } (1) \text{ for } 3^{2k}] \\ &= 72A + 9 - 1 \\ &= 72A + 8 \\ &= 8(9A + 1) \end{aligned}$$

This statement is a multiple of 8, as anything multiplied by 8 must also be divisible by that same number.

4. Conclusion: Because we have proven that the statement is a multiple of 8, we can conclude that the statement is true for all values of $n = k + 1$. Similarly, because the statement is true for all values of $n = k + 1$, it must also be true for all values of $n = k$. Therefore, the conjecture has been proven.

   $\therefore$ true for $n = k + 1$

   $\therefore$ true for $n = k$

   $\therefore$ conjecture is true.

PROBLEMS TO SOLVE

A conjecture is true only if it is always true. If it is false, give a counter example, otherwise give one example of when it is true.

1. The number 1 more than square of an even integer is always a prime number.
2. Every number has an even number of factors.
3. Every factor of an odd number is odd.
4. Every factor of an even number is even.
5. The product of two consecutive counting numbers is a multiple of 4.

Prove the following conjectures algebraically.

6. The sum of three consecutive even integers is always a multiple of three.
7. The sum of three counting numbers in an arithmetic progression is a multiple of three.
8. The product of three consecutive even numbers is always a multiple of eight.
9. If you multiply two odd numbers the result is always an odd number.
10. If x^2 is an odd number then x is an odd number.
11. Express the following recurring decimals as fully simplified fractions:
    - (a) $0.\dot{6}$
    - (b) $0.\dot{7}$
    - (c) $0.1\dot{6}$
    - (d) $0.\overline{09}$
    - (e) $0.41\dot{6}$
    - (f) $0.\overline{428571}$
12. Convert the following fractions to decimals, clearly indicating which (if any) digits are recurring:
    - (a) $\frac{5}{6}$
    - (b) $\frac{4}{9}$
    - (c) $\frac{1}{15}$
    - (d) $\frac{7}{30}$

(e) $\frac{9}{22}$

(f) $\frac{5}{13}$

13. Prove that $\sqrt{3}$ is irrational.

14. Prove that $\sqrt{5}$ is irrational.

15. Prove that $\log_2 5$ is irrational.

16. Prove that $\log_3 7$ is irrational.

17. Use mathematical induction to prove that for all positive integers n:

(a) $2 + 4 + 6 + \ldots + 2n = n(n + 1)$

(b) $1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n + 1)$

(c) $1 + 4 + 7 + \ldots + (3n - 2) = \frac{2(3n - 1)}{2}$

(d) $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$

(e) $1 \text{ x } 2 + 2 \text{ x } 3 + 3 \text{ x } 4 + 4 \text{ x } 5 + \ldots + n(n + 1) = \frac{1}{3}n(n + 1)(n + 2)$

(f) $\frac{1}{1x4} + \frac{1}{4x7} + \frac{1}{7x10} + \ldots + \frac{1}{(3n - 2)(3n + 1)} = \frac{n}{3n + 1}$

18. Use mathematical induction to prove the following divisibility tests for all positive integers n:

(a) $9^n - 3$ is a multiple of 6

(b) $3^{4n} - 1$ is divisible by 80

(c) $5^n + 3$ is divisible by 4

(d) $n(n^2 + 2)$ is divisible 3

(e) $5^{2n} - 1$ is a multiple of 24

(f) $n^3 - n$ is a multiple of 6, n is a positive integer $n \geq 2$

19. Using mathematical induction, prove that for all $n \geq 1$

$\frac{x^{n+1} - 1}{x - 1} = 1 + x + x^2 + \ldots + x^n$ where $x \neq 1$.

20. Using mathematical induction, show that for any positive integer n,

$a + ar + ar^2 + \ldots + ar^n = \frac{a(r^{n+1} - 1)}{r - 1}$ where $r \neq 1$.