Macmillan

Australian Curriculum

# MathsWorld 10

## Australian Curriculum edition

Covers Year 10 and 10A content descriptors


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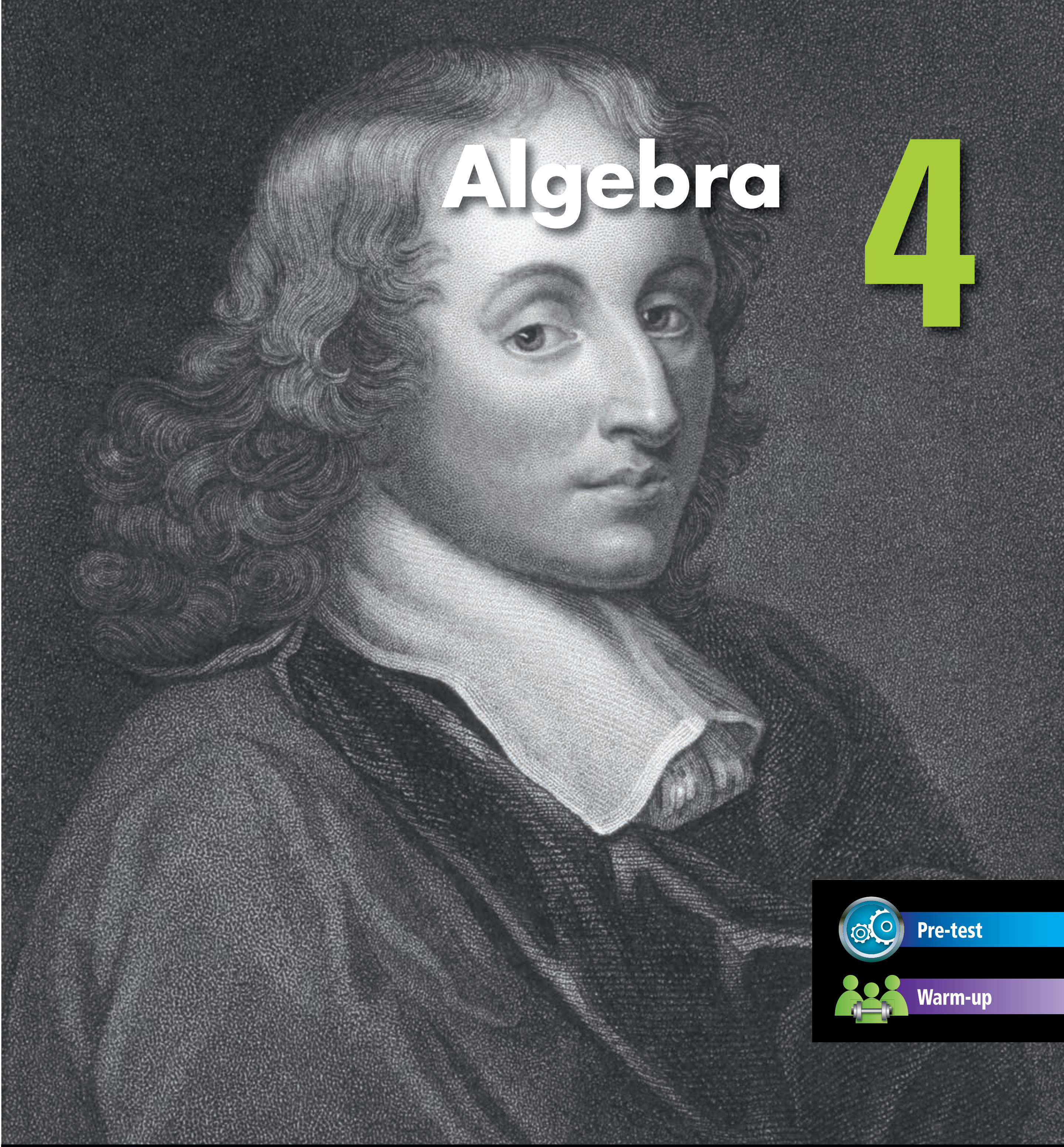
# 4 Algebra



Pre-test

Warm-up

Blaise Pascal was a French mathematician, physicist and philosopher. He was born in 1623 and died in 1662. During his short life he did important work in algebra, geometry, probability and physics. Pascal explored the coefficients of terms in binomial expansions of the type $(a + b)^n$. In this chapter, you will work with binomial expansions of the type $(a + b)^2$ but the analysis task at the end of the chapter will give you an opportunity to explore other binomial expansions.

# 4.1 Reviewing expanding, factorising and simplifying

## Expansion

Equivalent expressions give the same result for every value of the pronumerals. For example, $3(x + 5)$ is equivalent to $3x + 15$ for any value of $x$. We refer to $3(x + 5)$ as the **factorised form** of the expression and $3x + 15$ as the **expanded form**.

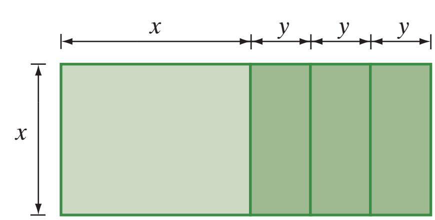
The process of going from the factorised form to the expanded form of an expression is called **expansion**.

**The distributive law**

In general, we can say that $a(b + c) = ab + ac$. This is called the **distributive law**.

### Example 1

Write expressions for the total area of the rectangle in factorised form and in expanded form.




**Working**

$x(x + 3y) = x^2 + 3xy$

**Reasoning**

The width and length of the rectangle are $x$ and $(x + 3y)$.
Applying the distributive law to $x(x + 3y)$:

$x(x + 3y)$

When we expand a factorised expression it is usual to simplify the expanded form, where possible.

**Example 2**

Expand each of the following, and simplify where possible.

**a** $a^2b(2a + 3b^2)$ **b** $2x(3x - 4) + 3x(x + 5)$ **c** $c(2c - 7) - (4c - 1)$

| Working | Reasoning |
| --- | --- |
| **a** $a^2b(2a + 3b^2)$<br>$= 2a^3b + 3a^2b^3$ | Multiply each term inside the brackets by $a^2b$. Remember that you can add indices when multiplying. |
| **b** $2x(3x - 4) + 3x(x + 5)$<br>$= 6x^2 - 8x + 3x^2 + 15x$<br>$= 9x^2 + 7x$ | Multiply each term in the first set of brackets by $2x$ and each term in the second set of brackets by $3x$. Simplify by grouping like terms. |
| **c** $c(2c - 7) - (4c - 1)$<br>$= 2c^2 - 7c - 4c + 1$<br>$= 2c^2 - 11c + 1$ | Expand the brackets using the distributive law.<br>Note that $-(4c - 1) = -4c - (-1)$.<br>$= -4c + 1$<br>Alternatively you could write $-(4c - 1) = -1(4c - 1)$ and expand by multiplying each term in the brackets by $-1$.<br>Simplify by grouping like terms. |

## Factorisation: taking out a common factor

The expression $3(x + 5)$ is the factorised form of the expression $3x + 15$ because it is written as the product of the **factors** 3 and $x + 5$. The process of going from the expanded form to the factorised form is called **factorisation**.

To factorise $3x + 15$, the **highest common factor** (or **HCF**) of $3x$ and 15 (that is, 3) is taken out in front of brackets to give $3(x + 5)$.

**Example 3**

Find the highest common factor of

**a** $10x$ and $5x^2$ **b** $2ab$ and $6a^2b$ **c** $2x^3$ and $4x^2$ **d** $8xy$ and $-18x^2z$

| Working | Reasoning |
| --- | --- |
| **a** $10x$ and $5x^2$<br>HCF $= 5x$ | HCF of 10 and 5 is 5.<br>HCF of $x$ and $x^2$ is $x$. |

continued

## Example 3 continued

**Working**

**b** $2ab$ and $6a^2b$
HCF $= 2ab$

**c** $2x^3$ and $4x^2$
HCF $= 2x^2$

**d** $8xy$ and $-18x^2z$
HCF $= 2x$

**Reasoning**

**b** HCF of 2 and 6 is 2.
HCF of $a$ and $a^2$ is $a$.
$b$ is a factor of both terms.

**c** HCF of 2 and 4 is 2.
HCF of $x^3$ and $x^2$ is $x^2$.

**d** HCF of 8 and $-18$ is 2.
HCF of $x$ and $x^2$ is $x$.
$y$ is not a factor of the second term.
$z$ is not a factor of the first term.

## Example 4

Factorise $12x^2 - 18xy$.

**Working**

$12x^2 - 18xy$
$= 6x(2x - 3y)$

**Reasoning**

The HCF of $12x^2$ and $-18xy$ is $6x$, so $6x$ goes in front of the bracket.

To find the terms inside the brackets, think:
$6x \times ? = 12x^2$ and $6x \times ? = -18xy$.

When both terms are negative, and usually when the first term is negative, a negative common factor is taken out.

## Example 5

Factorise.

**a** $-8ab - 18a^2b$

**b** $-6xy^2 + 21x^2y$

**Working**

**a** $-8ab - 18a^2b$
$= -2ab(4 + 9a)$

**b** $-6xy^2 + 21x^2y$
$= -3xy(2y - 7x)$

**Reasoning**

**a** When the negative factor is taken out, the subtraction sign between the terms becomes an addition sign inside the bracket.

**b** When the negative factor is taken out, the addition sign between the terms becomes a subtraction sign inside the bracket.

## Multiplying algebraic fractions

The product of algebraic fractions can be simplified by cancelling factors that are common to the numerator and denominator.

### Example 6

Simplify the following.

**a** $\dfrac{3ab}{12a^2}$

**b** $\dfrac{-8xy^3}{3x} \times \dfrac{-12x}{y}$

| Working | Reasoning |
|---|---|
| **a** $\dfrac{3ab}{12a^2} = \dfrac{b}{4a}$ | The highest common factor of the numerator and denominator is $3a$. |
| **b** $\dfrac{-8xy^3}{3x} \times \dfrac{-12x}{y}$ $= \dfrac{8 \times 12 \times x^2y^3}{3xy}$ $= 32xy^2$ | Simplify the numerator and simplify the denominator. The product of the two negative fractions is positive. The highest common factor of the numerator and denominator is $3xy$. |

## Adding and subtracting algebraic fractions

Algebraic fractions can be added and subtracted in the same way as we add and subtract number fractions.

### Example 7

Express as single fractions.

**a** $\dfrac{4x}{5} - \dfrac{x}{5}$

**b** $\dfrac{4a}{3} + \dfrac{5a}{3}$

**c** $\dfrac{4x}{9} + \dfrac{x}{4} + \dfrac{2x}{3}$

| Working | Reasoning |
|---|---|
| **a** $\dfrac{4x}{5} - \dfrac{x}{5}$ $= \dfrac{3x}{5}$ | The denominators are the same. Subtract $x$ from $4x$ to give the numerator. |
| **b** $\dfrac{4a}{3} + \dfrac{5a}{3}$ $= \dfrac{9a}{3}$ $= 3a$ | The denominators are the same. Add $4a$ and $5a$ to give the numerator. Divide the numerator and denominator by the common factor 3. |

continued

## Example 7 continued

**Working**

c $\dfrac{4x}{9} + \dfrac{x}{4} + \dfrac{2x}{3}$

$= \dfrac{4x}{9} \times \dfrac{4}{4} + \dfrac{x}{4} \times \dfrac{9}{9} + \dfrac{2x}{3} \times \dfrac{12}{12}$

$= \dfrac{16x}{36} + \dfrac{9x}{36} + \dfrac{24x}{36}$

$= \dfrac{49x}{36}$

**Reasoning**

The lowest common denominator is 36.
Express the fractions as equivalent fractions with the common denominator 36.
Add the numerators. The denominator is 36.

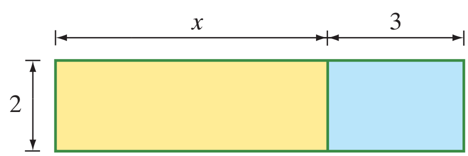
# exercise 4.1

LINKS TO Example 1

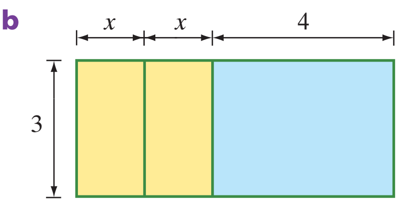
**1** Find the area of each of the following rectangles, giving your answer in

**i** factorised form **ii** expanded form.

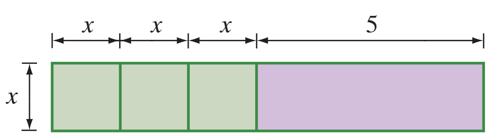
**a**



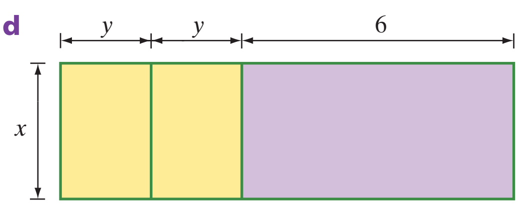

**b**




**c**




**d**




LINKS TO Example 2

**2** Expand each of the following, and simplify where possible.

**a** $2(x + 4)$ **b** $3(a - 5)$ **c** $4(2b - 1)$

**d** $6(3m + 2)$ **e** $-3(x + 2)$ **f** $-a(a + 3)$

**g** $-4(b - 5)$ **h** $-2p(7 - p^2)$ **i** $2(a + 1) + 3(a + 2)$

**j** $7(x - 5) + 2(x + 3)$ **k** $3(x + 9) - 5(2 + x)$ **l** $5(3 - 2a) - 4(6 - 3a)$

**m** $k(k + 7) + 2(k + 2)$ **n** $x(x + 3) + 5(x - 4)$ **o** $m(m + 7) - 3(m + 5)$

**3** Expand each of the following, and simplify where possible.

**a** $d(d - 2) - 3(2d + 5)$ **b** $g(3g - 8) - 2(g - 4)$ **c** $p(5 + 2p) - 3(4 - p)$

**d** $x(2x + 3) - 5(x - 4)$ **e** $2x(x - 7) - 3(5x - 2)$ **f** $3x(x - 4) - 5(2x + 3)$

LINKS TO Example 3

**4** Find the highest common factor of these pairs of terms.

**a** $2a^2b$ and $4a$ **b** $10xy$ and $15x^2$ **c** $6ab$ and $2b^2$

**d** $-32m^2n^3$ and $-8mn^2$ **e** $-14x^2y^3$ and $-42xy$ **f** $-18ab$ and $-45a^2c$

LINKS TO Example 4, 5

**5** Factorise each of the following.

a $6b + 6$
b $5a - 25b$
c $4ab - 12b^2$
d $pq - p^2q^2$
e $2xy - 4x^2y^2$
f $6ab - 8a^2b$
g $4xy - 12x^2$
h $6x^2y - 10xy^2$
i $7m^2n^2 + 28mn$
j $12a^2b + 16a^2b^2$
k $5xy - 10x^2y^2$
l $9p^2q - 12pq$
m $10b^3 - 6a^2$
n $5x^2 + 15x^4$
o $m^2n + m^2n^2$
p $20a^2b^2 - 10ab^3$
q $-4a - 16b$
r $-7a^2b^3 - 35a^4b^2$
s $-8x^2y - 12xy^2$
t $-18xy^3 + 30x^3$
u $-48m^2n^2 + 54mn^3$

LINKS TO Example 6

**6**
a $12a \div 3b$
b $8x \div 2x$
c $8x \div (-12y)$
d $6ab \div 8ac$
e $-15ab \div 10bc$
f $20abc \div 14ac$
g $(-100de) \div (-80efg)$
h $9xyz \div 12yz$
i $\dfrac{4xy}{6xz} \times \dfrac{10xy}{2x}$
j $\dfrac{4xy}{-6x} \times \dfrac{5xy}{2x}$
k $\dfrac{xy}{6z} \times \dfrac{-10xy}{-2x}$
l $\dfrac{4xy}{-6} \times \dfrac{-12xy}{15y}$
m $\dfrac{2abc}{3ad} \times \dfrac{-5ab}{2b}$
n $\dfrac{7fg}{-6g} \times \dfrac{3fx}{-2x}$
o $-\dfrac{3mn}{5mp} \times \dfrac{-10x}{2m}$

**7** $\dfrac{6a^2b}{12a}$ simplifies to

A $2ab$
B $\dfrac{ab}{2}$
C $6ab$
D $\dfrac{ab}{6}$
E $\dfrac{b}{2a}$

LINKS TO Example 7

**8** Express as single fractions.

a $\dfrac{3x}{7} + \dfrac{2x}{7}$
b $\dfrac{8x}{11} - \dfrac{3x}{11}$
c $\dfrac{4x}{9} - \dfrac{x}{9} + \dfrac{2x}{9}$
d $-\dfrac{a}{6} + \dfrac{5a}{6} - \dfrac{7a}{6}$
e $\dfrac{3a}{5} + \dfrac{a}{3}$
f $\dfrac{5x}{8} + \dfrac{7x}{16}$
g $\dfrac{3x}{4} + \dfrac{x}{2} - \dfrac{4x}{3}$
h $\dfrac{3a}{4} + \dfrac{a}{3} - \dfrac{5a}{6}$
i $\dfrac{7x}{5} - \dfrac{3x}{4} + \dfrac{9x}{10}$

## exercise 4.1 challenge

**9** a Find the values of the pronumerals $a$, $b$, $c$ and $d$ that make the following identity true:
$3x^2(x + 2)(x - 3) = ax^4 + bx^3 + cx^2 + dx + e$.

b Find the values of the pronumerals $a$ and $b$ that make the following identity true:
$x(x - a) + 2(x + 2) = x^2 - x + b$.

# 4.2 Binomial expansion

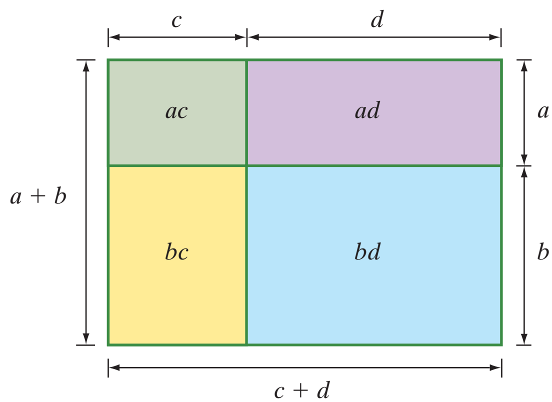
A binomial expression is an expression made up of two terms. For example, $(x + 2)$ and $(2a - 3)$ are binomial expressions.

The following diagram illustrates the expansion of $(a + b)(c + d)$.

Binomial expansion

**Binomial expansion**

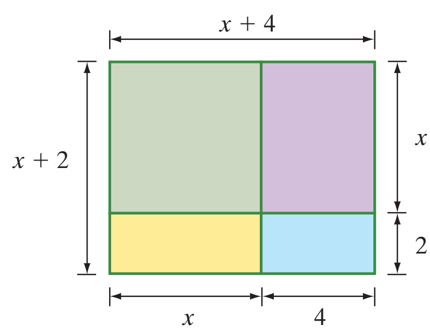
$(a + b)(c + d) = ac + ad + bc + bd$




Binomial expansions: a matching activity

## Example 8

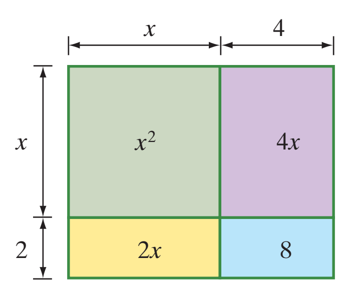
Use this diagram to find the expansion of $(x + 2)(x + 4)$.




**Working**

$$\begin{aligned}(x + 2)(x + 4) &= x^2 + 4x + 2x + 8 \\ &= x^2 + 6x + 8\end{aligned}$$

**Reasoning**



Algebraically, expressions like the previous example $(x + 2)(x + 4)$ are expanded by multiplying the whole of the second bracket by each term in the first bracket, which results in four terms. Sometimes, there are **like terms** that can be gathered.

**Using FOIL to expand**

To help remember the four expansion terms, you can use the word FOIL, which stands for

- **F**irst: find the product of the first term in each bracket.
- **O**utside: find the product of the first term in the first bracket and the last term in the second bracket.
- **I**nside: find the product of the last term in the first bracket and the first term in the second bracket.
- **L**ast: find the product of the last term in each bracket.

## Example 9

Expand each of the following, and simplify where possible.

**a** $(x - 3)(x + 7)$ **b** $(2a - 5)(a - 3)$ **c** $(a - 7)(b^2 + 4)$

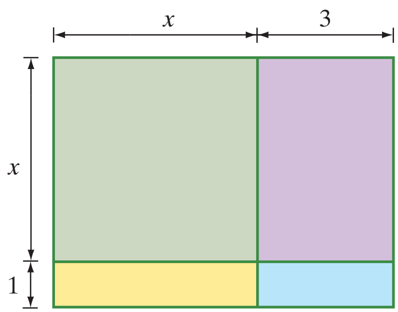
| **Working** | **Reasoning** |
|---|---|
| **a** $(x - 3)(x + 7)$<br>$= x^2 + 7x - 3x - 21$<br>$= x^2 + 4x - 21$ | Use FOIL and collect like terms.<br>$(x - 3)(x + 7)$ |
| **b** $(2a - 5)(a - 3)$<br>$= 2a^2 - 6a - 5a + 15$<br>$= 2a^2 - 11a + 15$ | Use FOIL and collect like terms.<br>$(2a - 5)(a - 3)$<br>The last term is positive because $-5 \times (-3) = 15$. |
| **c** $(a - 7)(b^2 + 4)$<br>$= ab^2 + 4a - 7b^2 - 28$ | Use FOIL.<br>$(a - 7)(b^2 + 4)$ |

## exercise 4.2

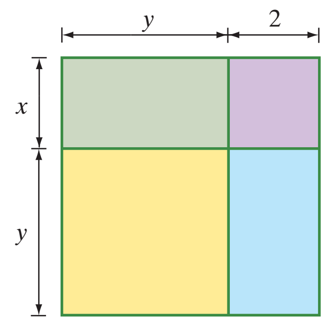
LINKS TO Example 8

**1** Use each diagram to write an algebraic expression in factorised form and in expanded form.

**a**




**b**




LINKS TO Example 9

**2** Expand and simplify each of the following.

**a** $(a + 4)(a + 2)$ **b** $(b + 3)(b + 5)$ **c** $(c - 3)(c + 7)$

**d** $(d + 1)(d - 4)$ **e** $(e - 6)(e - 2)$ **f** $(f - 5)(f - 3)$

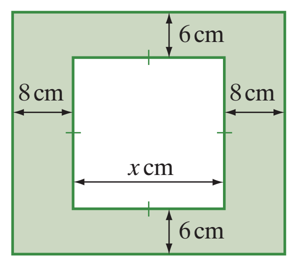
**g** $(2g + 3)(g + 5)$ **h** $(3h - 2)(2h - 1)$ **i** $(x + 3)(2x + 1)$

**j** $(a - 8)(3a + 4)$ **k** $(m - 7)(4m + 1)$ **l** $(2y - 3)(3y + 2)$

**m** $(3a - 4b)(a + 2b)$ **n** $(m - 3n)(5m - 6n)$ **o** $(2x - 1)(3x + 5)$

**p** $(4 - x)(5x + 2)$ **q** $(7 - a)(4 - 3a)$ **r** $(2a - 5b)(3b - 4a)$

**3** A square mirror is surrounded by a frame, as shown in the diagram.




**a** Write down expressions for the length and width of the framed mirror in terms of $x$.

**b** Write down an expression for the area of the framed mirror in expanded form.

**c** Find the area of the framed mirror when $x = 40$.

## exercise 4.2 challenge

**4** Find the lowest common multiple of $5x^2 - 25xy$ and $10x^2y^2 + 15xy^3$. Express your answer in both factorised form and expanded form.

**5** Expand and simplify each of the following.

**a** $(x + 1)(x + 2)(x + 3)$ **b** $2(x - 4)(x + 1)(x - 2)$

# 4.3 Common binomial factors

Some algebraic expressions have binomial expressions, rather than single terms, as factors.

In the expression $x(x - 1) + 2(x - 1)$, the common factor is $x - 1$. As shown in the following example, this term can be taken out like any other common factor.

## Example 10

Factorise each of the following.

**a** $x(x - 1) + 2(x - 1)$

**b** $p(2p - 3) - 2(2p - 3)$

**c** $a(a - 3) + a - 3$

**d** $16xy(y + 3) - 24y(y + 3)$

| Working | Reasoning |
|---|---|
| **a** $x(x - 1) + 2(x - 1)$ <br> $= (x - 1)(x + 2)$ | The common factor of $x - 1$ is taken out in front of a bracket. |
| **b** $p(2p - 3) - 2(2p - 3)$ <br> $= (2p - 3)(p - 2)$ | The common factor of $2p - 3$ is taken out in front of a bracket. |
| **c** $a(a - 3) + \mathrm{a} - 3$ <br> $= a(a - 3) + 1(a - 3)$ <br> $= (a - 3)(a + 1)$ | Writing $+ a - 3$ as $+ 1(a - 3)$ makes it easier to see that $a - 3$ is a common factor. |
| **d** $16xy(y + 3) - 24y(y + 3)$ <br> $= 8y(y + 3) \times 2x + 8y(y + 3) \times -3$ <br> $= 8y(y + 3)(2x - 3)$ | We see that $y + 3$ is a common factor. Also, $8y$ is a common factor of $16xy$ and $-24y$. So $8y(y + 3)$ is taken out in front of a bracket. |

Sometimes the order of two terms may need to be changed so that the terms are in the same order in both binomial factors.

## Example 11

Factorise each of the following.

**a** $2x(x + 3) + 5(3 + x)$

**b** $3m(2m - 3) + 6 - 4m$

| Working | Reasoning |
|---|---|
| **a** $2x(x + 3) + 5(3 + x)$ <br> $= 2x(x + 3) + 5(x + 3)$ <br> $= (x + 3)(2x + 5)$ | When adding two numbers, the order doesn't matter, so $3 + x = x + 3$. The common factor $x + 3$ is taken out in front of a bracket. |

continued

**Example 11** continued

| Working | Reasoning |
|---|---|
| **b** $3m(2m - 3) + 6 - 4m$<br>$= 3m(2m - 3) + 2(3 - 2m)$<br>$= 3m(2m - 3) - 2(2m - 3)$ | The expression $6 - 4m$ has a common factor of 2.<br>When subtracting, order does matter: $(3 - 2m)$ is not the same as $(2m - 3)$.<br>Use $2(3 - 2m) = -2(2m - 3)$ |
| $= (2m - 3)(3m - 2)$ | The common factor $2m - 3$ is taken out in front of a bracket. |

# Factorisation by grouping 'two and two'

Consider the expression $ax + ay + 2x + 2y$. We can group the terms in pairs so that each pair has a common factor.

$ax + ay + 2x + 2y = a(x + y) + 2(x + y)$

We can then take out the binomial factor $x + y$ to show that

$ax + ay + 2x + 2y = (x + y)(a + 2)$

Sometimes we first need to rearrange the terms in order to group 'two and two'.

> Is there another way of grouping the four terms in pairs so that each pair has a common factor?

**Example 12**

To factorise the following expressions, rearrange the terms if necessary and then group them in pairs.

**a** $at + aq + 4t + 4q$ **b** $p^2 - 2pq - 4p + 8q$ **c** $3ab - 2c + ac - 6b$

| Working | Reasoning |
|---|---|
| **a** $at + aq + 4t + 4q$<br>$= a(t + q) + 4(t + q)$ | $a$ is a common factor of the first pair of terms and 4 is a common factor of the second pair of terms. |
| $= (t + q)(a + 4)$ | The common factor $t + q$ is taken out in front of a bracket. |
| **b** $p^2 - 2pq - 4p + 8q$<br>$= p(p - 2q) - 4(p - 2q)$ | $p$ is a common factor of the first pair of terms. Instead of taking out 4 as a factor of $-4p + 8q$, we take out $-4$ and obtain the same binomial common factor, $p - 2q$. |
| $= (p - 2q)(p - 4)$ | $p - 2q$ is taken out in front of a bracket. |

continued

## Example 12 continued

| Working | Reasoning |
|---|---|
| **c** $3ab - 2c + ac - 6b$<br>$= 3ab - 6b + ac - 2c$<br>$= 3b(a - 2) + c(a - 2)$ | Rearrange the terms so that each pair of terms has a common factor.<br>$3b$ is a common factor of the first pair of terms and $c$ is a common factor of the second pair of terms. |
| $= (a - 2)(3b + c)$ | The common factor $(a - 2)$ is taken out in front of a bracket. |
| Alternatively, $3ab - 2c + ac - 6b$<br>$= 3ab + ac - 6b - 2c$<br>$= a(3b + c) - 2(3b + c)$<br>$= (3b + c)(a - 2)$ | The four terms can be rearranged in a different way to give $(3b + c)$ as the common factor of each pair of terms. |

## Example 13

Factorise each of the following.

**a** $(m + 5)^2 + 3(m + 5)$

**b** $6(x - 3) - (x - 3)^2$

| Working | Reasoning |
|---|---|
| **a** $(m + 5)^2 + 3(m + 5)$<br>$= (m + 5)(m + 5) + 3(m + 5)$ | There is a common factor of $m + 5$. |
| $= (m + 5)(m + 5 + 3)$ | $m + 5$ is taken out in front of a bracket. |
| $= (m + 5)(m + 8)$ | Simplify the second bracket. |
| **b** $6(x - 3) - (x - 3)^2$<br>$= 6(x - 3) - (x - 3)(x - 3)$ | There is a common factor of $x - 3$. |
| $= (x - 3)[6 - (x - 3)]$ | $x - 3$ is taken out in front of a bracket. |
| | Remember to use brackets when subtracting a binomial expression. |
| $= (x - 3)(6 - x + 3)$<br>$= (x - 3)(9 - x)$ | Simplify the second bracket. |

## exercise 4.3

LINKS TO Example 10

**1** Factorise each of the following by taking out a binomial factor.

**a** $a(x + 1) + 3(x + 1)$
**b** $4(x - 7) + x(x - 7)$
**c** $p(2p + 3) - 2(2p + 3)$
**d** $y(y - 6) - (y - 6)$
**e** $2w(3w - 5) + 6(3w - 5)$
**f** $5(3 - 4z) - 8z(3 - 4z)$
**g** $a^2(b - 2c) + b - 2c$
**h** $x(y - z) - y + z$
**i** $a(a + 3) + a + 3$
**j** $p(p - 1) - p + 1$
**k** $x(x + 5) + 5(x + 5)$
**l** $2m(2m + 3) + 3(2m + 3)$

**2** Fill in the gaps to factorise by grouping the terms in pairs.

**a** $ax + ay + 3x + 3y$
$= \_\_(x + y) + \_\_(x + y)$
$= (x + y)(\_\_\_\_)$

**b** $x^2 - x + yx - y$
$= \_\_(x - 1) + \_\_(x - 1)$
$= (x - 1)(\_\_\_\_)$

**c** $3m + 18 + am + 6a$
$= \_\_(\_\_\_\_) + \_\_(\_\_\_\_)$
$= (\_\_\_\_)(\_\_\_\_)$

**d** $yz + 7z - 2y - 14$
$= z(\_\_\_\_) - 2(\_\_\_\_)$
$= (\_\_\_\_)(\_\_\_\_)$

**e** $3ab + 6b + 2a + 4$
$= \_\_(\_\_\_\_) + \_\_(\_\_\_\_)$
$= (\_\_\_\_)(\_\_\_\_)$

**f** $4k - 16 + km - 4m$
$= \_\_(\_\_\_\_) + \_\_(\_\_\_\_)$
$= (\_\_\_\_)(\_\_\_\_)$

LINKS TO Example 11

**3** Factorise each of the following expressions.

**a** $x(x + 1) + 5(1 + x)$
**b** $a(a + 4) - 3(4 + a)$
**c** $7(x - 5) + x(x - 5)$
**d** $a(a - 3) + 4(3 - a)$
**e** $m(2m - 7) + 1(7 - 2m)$
**f** $2x(x - 9) - 3(9 - x)$
**g** $3a(a - 4) + 2(4 - a)$
**h** $5m(6 - 5m) - 3(5m - 6)$
**i** $3x(x - 3) + 6 - 2x$
**j** $4a(2a - 5) + 15 - 6a$
**k** $2p(3p - 4) + 20 - 15p$
**l** $3x(7 - 4x) + 8x - 14$

**4** Factorise by first grouping the terms in pairs.

**a** $ab + ac + eb + ec$
**b** $tp + tq + 3p + 3q$
**c** $5x + 5y + kx + ky$
**d** $6st + 12t + 3s + 6$
**e** $a^2 - 3ab + 4a - 12b$
**f** $3x + 9 - 2xy - 6y$
**g** $5p + 15 - 3pq - 9q$
**h** $6a + 12 - 2ab - 4b$
**i** $a^2 - 3ab - 3a + 9b$
**j** $8a - 2 - 8ab + 2b$
**k** $a^2 - 4a + 2a - 8$
**l** $x^2 + 7x - 2x - 14$

LINKS TO Example 12

**5** Factorise by first rearranging the terms if necessary, and then grouping them in pairs.

**a** $x^2 + 6y + xy + 6x$
**b** $4a - 3b + ab - 12$
**c** $2ap + 6 + 4a + 3p$
**d** $xy - 8 + 8y - x$
**e** $pq - 16r^2 + pr - 16qr$
**f** $p^2 - a - p + ap$
**g** $6m - 3n - 18 + mn$
**h** $1 - xy + x - y$
**i** $xy + 5z + 5x + yz$
**j** $ab + cd + ac + bd$
**k** $5m + 3n + mn + 15$
**l** $7x - 3y + xy - 21$

LINKS TO Example 13

**6** Factorise each of the following.

**a** $5(x + 2) + (x + 2)^2$
**b** $(c + 3)^2 + (c + 3)$
**c** $(a - 4)^2 - 2(a - 4)$
**d** $3(r + 7) + (r + 7)^2$
**e** $(x - 4)^2 - 5(x - 4)$
**f** $5(y + 8) + (y + 8)^2$
**g** $9(c + 2) - (c + 2)^2$
**h** $3(p - 4) - (p - 4)^2$
**i** $2(x + 1)^2 - (x + 1)$
**j** $3(y - 4) - 6(y - 4)^2$
**k** $2(m + n)^2 - m - n$
**l** $2(x - 2)^2 + 3(2 - x)$
**m** $3(y - 4)^2 - 5(4 - y)$

# 4.4 Perfect squares

## Expanding a perfect square

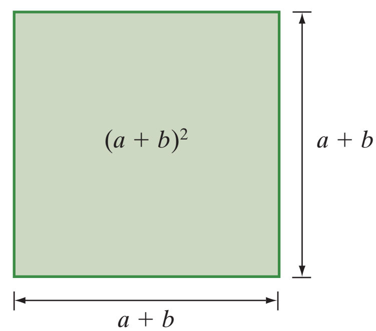
The expressions $(a + b)^2$ and $(a - b)^2$ are examples of **perfect squares**. To square a binomial expression, we multiply it by itself. For example, $(a + b)^2 = (a + b)(a + b)$. We could then use FOIL to help us expand $(a + b)(a + b)$ or similar squares.

The number patterns in these perfect squares can be used to expand a perfect square.

| Factorised form | Expanded form |
|---|---|
| $(x - 1)^2$ | $x^2 - 2x + 1$ |
| $(x - 2)^2$ | $x^2 - 4x + 4$ |
| $(x + 4)^2$ | $x^2 + 8x + 16$ |
| $(x + 7)^2$ | $x^2 + 14x + 49$ |

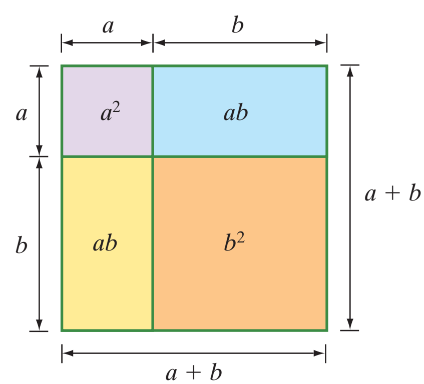
| Factorised form | Expanded form |
|---|---|
| $(2x - 1)^2$ | $4x^2 - 4x + 1$ |
| $(3x + 2)^2$ | $9x^2 + 12x + 4$ |
| $(x + 3)^2$ | $x^2 + 6x + 9$ |
| $(x - 5)^2$ | $x^2 - 10x + 25$ |

We can also use a diagram for $(a + b)^2$ to obtain a general rule that enables us to expand perfect squares quickly.



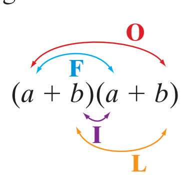

**Expanding a perfect square**
The square area $(a + b)^2$ is made up of
- a square with area $a^2$
- two rectangles, each with area $ab$
- another square, with area $b^2$.

So, $(a + b)^2 = a^2 + 2ab + b^2$.




This can also be shown algebraically using FOIL.

$$(a + b)^2 = (a + b)(a + b)$$
$$= a^2 + ab + ab + b^2$$
$$= a^2 + 2ab + b^2$$




The expression $(a - b)^2$ is also a perfect square.

$$(a - b)^2 = (a - b)(a - b)$$
$$= a^2 - ab - ab + b^2$$
$$= a^2 - 2ab + b^2$$

**Recognising a perfect square**

The following properties will help you to recognise a perfect square in expanded form.

- There must be three terms, as a perfect square is a **trinomial** expression.
- The first and last terms must both be *positive* and *perfect squares*: $a^2$ and $b^2$. (Check these terms first and identify $a$ and $b$.)
- The middle term must be *double the product* of $a$ and $b$ (and may be either positive or negative).

## Example 14

Which of the following are perfect squares? (Give your reasons.)

**a** $y^2 + 8y - 16$ **b** $m^2 - 6m + 9$ **c** $b^2 + 4bc + 16c^2$

**Working**

**a** Not a perfect square because the third term is negative, it needs to be $+\ 16$

**b** Perfect square because $9 = 3^2$ and $2 \times 3 = 6$

**c** Not a perfect square, because $16c^2 = (4c)^2$ but $4bc$ does not equal $2 \times 4c \times b$.

**Reasoning**

$y^2 + 8y + 16 = (y + 4)^2$

$m^2 - 6m + 9 = (m - 3)^2$

$b^2 + 8bc + 16c^2 = (b + 4c)^2$

## Example 15

Expand these perfect squares.

**a** $(x + 6)^2$ **b** $(3b - 2)^2$ **c** $(x - 3y)^2$

**Working**

**a** $(x + 6)^2 = x^2 + 2 \times x \times 6 + 6^2$
$= x^2 + 12x + 36$

**b** $(3b - 2)^2 = (3b)^2 - 2 \times 3b \times 2 + 2^2$
$= 9b^2 - 12b + 4$

**Reasoning**

Use $(a + b)^2 = a^2 + 2ab + b^2$, where $a$ is $x$ and $b$ is 6.

Use $(a - b)^2 = a^2 - 2ab + b^2$, where $a$ is $3b$ and $b$ is 2.

Remember to square both $b$ and the coefficient of $b$:
$(3b)^2 = 3b \times 3b = 9b^2$

continued

**Example 15** continued

| Working | Reasoning |
|---|---|
| **c** $(x + 3y)^2 = x^2 + 2 \times x \times (3y) + (3y)^2$<br>$= x^2 + 6xy + 9y^2$ | Use $(a + b)^2 = a^2 + 2ab + b^2$, where $a$ is $x$ and $b$ is $3y$.<br>Remember to square both $y$ and the coefficient of $y$: $(3y)^2 = 3y \times 3y = 9y^2$ |

# Factorising a perfect square

Going from factorised form to expanded form, $(a + b)^2 = a^2 + 2ab + b^2$. It follows that the reverse is also true, so $a^2 + 2ab + b^2 = (a + b)^2$. Similarly, $a^2 - 2ab + b^2 = (a - b)^2$. Therefore, if you can recognise a perfect square in expanded form, it is easy to factorise.

**Expanding or factorising perfect squares**

Expansion

$$(a + b)^2 = a^2 + 2ab + b^2$$

Factorisation

Expansion

$$(a - b)^2 = a^2 - 2ab + b^2$$

Factorisation

**Example 16**

Factorise each of the following.

**a** $x^2 + 14x + 49$ **b** $m^2 - 8m + 16$ **c** $14x + 49 + x^2$

| Working | Reasoning |
|---|---|
| **a** $x^2 + 14x + 49$<br>$= (x + 7)^2$ | $x^2 + 14x + 49$ is a perfect square of the form $a^2 + 2ab + b^2$, where $a$ is $x$ and $b$ is 7. |
| **b** $m^2 - 8m + 16$<br>$= (m - 4)^2$ | $m^2 - 8m + 16$ is a perfect square of the form $a^2 - 2ab + b^2 = (a - b)^2$, where $a$ is $m$ and $b$ is 4. |
| **c** $14x + 49 + x^2$<br>$= x^2 + 14x + 49$<br>$= (x + 7)^2$ | Rewrite $14x + 49 + x^2$ in the form $a^2 + 2ab + b^2$.<br>$a^2 + 2ab + b^2 = (a + b)^2$ |

Sometimes it is necessary to take out a common factor first. If the coefficient of the $x^2$ term is $-1$, then $-1$ should be taken out as a factor.

## Example 17

Factorise the following expressions.

**a** $2x^2 + 16x + 32$ **b** $-x^2 + 10x - 25$

| Working | Reasoning |
|---|---|
| **a** $2x^2 + 16x + 32$ | Take out a common factor of 2. |
| $= 2(x^2 + 8x + 16)$ | $x^2 + 8x + 16$ can be written in the form |
| $= 2(x^2 + 2 \times 4x + 4^2)$ | $a^2 + 2ab + b^2$, where $a = x$ and $b = 4$. |
| $= 2(x + 4)^2$ | $a^2 + 2ab + b^2 = (a + b)^2$ |
| **b** $-x^2 + 10x - 25$ | When the negative sign is taken outside the bracket, the signs of the terms inside the brackets change. |
| $= -(x^2 - 10x + 25)$ | $x^2 - 10x + 25$ can now be factorised as a perfect square. |
| $= -(x - 5)^2$ | |

In example 17 part a the expression has a common factor of 2. When this is taken out, the expression in the brackets can be seen to be a perfect square.

## exercise 4.4

LINKS TO Example 14

**1** Which of the following quadratic expressions are perfect squares? For each expression that is not a perfect square, show how it could be turned into a perfect square by changing one term.

**a** $x^2 - 12x - 36$ **b** $b^2 - 18x + 81$ **c** $h^2 + 7h + 49$

**d** $y^2 - 8y + 8$ **e** $m^2 + 2m + 4$ **f** $x^2 - 12x + 36$

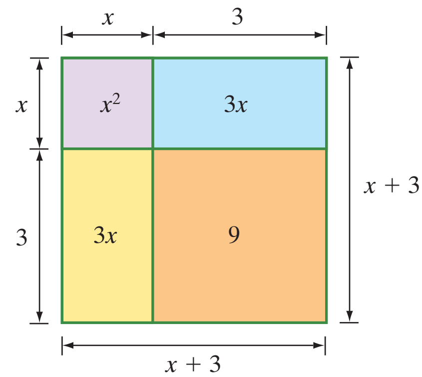
LINKS TO Example 15

**2** Copy and complete this table of perfect squares.

| Factorised form | Expanded form |
|---|---|
| $(x + 1)^2$ | |
| $(x + 2)^2$ | |
| $(x + 3)^2$ | |
| $(x + 4)^2$ | |
| $(x + 5)^2$ | |
| $(x + 6)^2$ | |

| Factorised form | Expanded form |
|---|---|
| $(x + 7)^2$ | |
| $(x + 8)^2$ | |
| $(x + 9)^2$ | |
| $(x + 10)^2$ | $x^2 + 20x + 100$ |
| $(x + 11)^2$ | |
| $(x + 12)^2$ | |

**3** Use the diagram to expand $(x + 3)^2$.




**4** **a** Draw a diagram that could be used to expand $(x + 4)^2$.

**b** Hence write down the expansion of $(x + 4)^2$.

**5** Expand each of the following.

| | | |
|---|---|---|
| **a** $(a + 6)^2$ | **b** $(m - 5)^2$ | **c** $(x + 3)(x + 3)$ |
| **d** $(c - 4)(c - 4)$ | **e** $(p + 5)^2$ | **f** $(m - 6)^2$ |
| **g** $(a + 2)(a + 2)$ | **h** $(b - 3)(b - 3)$ | **i** $(3 + 2x)(3 + 2x)$ |
| **j** $(4x - 3)^2$ | **k** $(x + 3y)^2$ | **l** $(5a - 2b)^2$ |
| **m** $(7 - 2k)^2$ | **n** $(8 - 3x)(8 - 3x)$ | **o** $(3x + 1)^2$ |
| **p** $(2 + 3x)(2 + 3x)$ | **q** $(2a + 5b)^2$ | **r** $(2 - 7y)^2$ |

LINKS TO Example 16

**6** Factorise each of the following perfect squares.

| | | |
|---|---|---|
| **a** $x^2 + 8x + 16$ | **b** $y^2 + 6y + 9$ | **c** $p^2 - 4p + 4$ |
| **d** $m^2 - 10m + 25$ | **e** $k^2 + 20k + 100$ | **f** $x^2 + 14x + 49$ |
| **g** $a^2 - 12a + 36$ | **h** $36 + 12x + x^2$ | **i** $p^2 + 4pq + 4q^2$ |
| **j** $16 - 8m + m^2$ | **k** $x^2 - 2x + 1$ | **l** $x^2 - 32x + 256$ |

LINKS TO Example 17

**7** Factorise each of the following.

| | | |
|---|---|---|
| **a** $x^2 + 14x + 49$ | **b** $y^2 + 8y + 16$ | **c** $p^2 - 6p + 9$ |
| **d** $k^2 - 10k + 25$ | **e** $z^2 + 22z + 121$ | **f** $64 + 16x + x^2$ |
| **g** $m^2 + 10m + 25$ | **h** $x^2 - 18x + 81$ | **i** $4 + 4x + x^2$ |
| **j** $24x + x^2 + 144$ | **k** $y^2 - 26y + 169$ | **l** $a^2 + 30a + 225$ |
| **m** $400 - 40x + x^2$ | **n** $36 + m^2 - 12m$ | **o** $x^2 + 3x + 2.25$ |

**8** Factorise each of the following.

| | | |
|---|---|---|
| **a** $2x^2 + 16x + 32$ | **b** $3x^2 + 6x + 3$ | **c** $3x^2 - 30x + 75$ |
| **d** $-x^2 - 4x - 4$ | **e** $-x^2 + 6x - 9$ | **f** $-x^2 + 14x - 49$ |
| **g** $-2x^2 + 36x - 162$ | **h** $-5x^2 - 30x - 45$ | |

## exercise 4.4 challenge

**9** A square has its length and width both decreased by 3 m. Let the side length of the square be $x$ cm.

**a** What is the area of the new square? **b** By how much has the area decreased?

# 4.5 Differences of two squares

Consider the expansion of $(a + b)(a - b)$, using FOIL.

$$(a + b)(a - b) = a^2 - ab + ab - b^2$$
$$= a^2 - b^2$$

$$(a + b)(a - b)$$

The result is a **difference of two squares**, because the expression is a subtraction or difference between two terms, each of which are squares.

**Expanding or factorising the difference of squares**

Expansion

$$(a + b)(a - b) = a^2 - b^2$$

Factorisation

## Example 18

Expand each of the following.

**a** $(x + 11)(x - 11)$

**b** $(2x - 3)(2x + 3)$

| Working | Reasoning |
|---|---|
| **a** $(x + 11)(x - 11)$<br>$= x^2 - 11^2$<br>$= x^2 - 121$ | Use $(a + b)(a - b) = a^2 - b^2$, where $a$ is $x$ and $b$ is 11. |
| **b** $(2x - 3)(2x + 3)$<br>$= (2x)^2 - 3^2$<br>$= 4x^2 - 9$ | Use $(a - b)(a + b) = a^2 - b^2$, where $a$ is $2x$ and $b$ is 3.<br>Simplify each square. |

In example 19 part b, the difference of squares includes the square of a binomial expression.

## Example 19

Factorise each of the following.

**a** $m^2 - 16$

**b** $(x + 1)^2 - 9$

| Working | Reasoning |
|---|---|
| **a** $m^2 - 16 = m^2 - 4^2$<br>$= (m - 4)(m + 4)$ | Write the expression in the form $a^2 - b^2$.<br>Use $a^2 - b^2 = (a - b)(a + b)$. |

continued

**Example 19** continued

**Working**

**b** $(x + 1)^2 - 9 = (x + 1)^2 - 3^2$
$= (x + 1 + 3)(x + 1 - 3)$
$= (x + 4)(x - 2)$

**Reasoning**

$(x + 1)^2 - 9$ is in the form $a^2 - b^2$, where $a$ is $x + 1$ and $b$ is 3.
Simplify each bracket.

In example 20 part a, each term has a coefficient that is a square number. We can write $4g^2$ as $(2g)^2$ and $25h^2$ as $(5h)^2$ so the expression is a difference of squares.

**Example 20**

Factorise each of the following expressions.

**a** $4g^2 - 25h^2$ **b** $3x^2 - 27y^2$

**Working**

**b** $4g^2 - 25h^2 = (2g)^2 - (5h)^2$
$= (2g - 5h)(2g + 5h)$

**c** $3x^2 - 27y^2 = 3(x^2 - 9y^2)$
$= 3[x^2 - (3y)^2]$
$= 3(x - 3y)(x + 3y)$

**Reasoning**

Write the expression in the form $a^2 - b^2$.
Use $a^2 - b^2 = (a - b)(a + b)$.

Take out the common factor of 3.
Write the expression inside the brackets in the form $a^2 - b^2$.
Use $a^2 - b^2 = (a - b)(a + b)$.

## Factorisation over *R*

So far, all the expressions that have been factorised have involved only rational numbers. We refer to this as **factorisation over *Q***, the rational number field. Some expressions cannot be factorised using only rational numbers, but they can be factorised using surds. We refer to this as **factorisation over *R***, the real number field.

The expression $x^2 - 5$, for example, cannot be factorised over $Q$, since 5 is not a perfect square. However, this expression can be factorised over $R$ using the difference of squares rule.

$$x^2 - 5 = x^2 - (\sqrt{5})^2$$
$$= (x - \sqrt{5})(x + \sqrt{5})$$

**Example 21**

Factorise the following expressions over $R$.

**a** $a^2 - 3$ **b** $2p^2 - 16$ **c** $(x + 1)^2 - 7$

**Working**

**a** $a^2 - 3 = a^2 - (\sqrt{3})^2$
$= (a - \sqrt{3})(a + \sqrt{3})$

**Reasoning**

Use the difference of squares rule $a^2 - b^2 = (a - b)(a + b)$, where $b^2 = 3$, that is, $b$ is $\sqrt{3}$.

continued

**Example 21** continued

| Working | Reasoning |
|---|---|
| **b** $2p^2 - 16 = 2(p^2 - 8)$ | First take out the common factor. |
| $= 2[p^2 - (\sqrt{8})^2]$ | Then use $a^2 - b^2 = (a - b)(a + b)$, |
| $= 2(p + \sqrt{8})(p - \sqrt{8})$ | where $a$ is $p$ and $b$ is $\sqrt{8}$. |
| $= 2(p + 2\sqrt{2})(p - 2\sqrt{2})$ | $\sqrt{8}$ can be simplified: $\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}$ |
| **c** $(x + 1)^2 - 7 = (x + 1)^2 - (\sqrt{7})^2$ | Use $a^2 - b^2 = (a - b)(a + b)$, |
| $= (x + 1 - \sqrt{7})(x + 1 + \sqrt{7})$ | where $a$ is $x + 1$ and $b$ is $\sqrt{7}$. |

## exercise 4.5

LINKS TO Example 18

**1** Use the 'difference of two squares' rule to expand each of the following.

**a** $(p - 4)(p + 4)$ **b** $(x + 5)(x - 5)$ **c** $(m - n)(m + n)$
**d** $(9 - m)(9 + m)$ **e** $(2x - 1)(2x + 1)$ **f** $(3 + 2x)(3 - 2x)$
**g** $(9 + 5y)(9 - 5y)$ **h** $(2a - 7b)(2a + 7b)$ **i** $(ab - 4)(ab + 4)$
**j** $(p - 7)(p + 7)$ **k** $(8 - m)(8 + m)$ **l** $(a - c)(a + c)$
**m** $(2x - 3)(2x + 3)$ **n** $(7 + 5k)(7 - 5k)$ **o** $(5 + 2a)(5 - 2a)$
**p** $(2m + 7)(2m - 7)$ **q** $(5 - 3x)(5 + 3x)$ **r** $(3 - 4a)(3 + 4a)$

**2** Copy and complete this table of differences of two squares.

| Factorised form | Expanded form |
|---|---|
| $(x + y)(x - y)$ | |
| | $m^2 - n^2$ |
| $(y + 3)(y - 3)$ | |
| | $x^2 - 4$ |
| $(a + 2b)(a - 2b)$ | |
| | $9x^2 - 4y^2$ |

| Factorised form | Expanded form |
|---|---|
| $(y + 5)(y - 5)$ | |
| $(7 + a)(7 - a)$ | |
| $(2y - 5x)(2y + 5x)$ | |
| | $r^2 - t^2$ |
| | $y^2 - 64$ |
| | $25x^2 - 4y^2$ |

LINKS TO Example 19a

**3** Use the 'difference of two squares' rule to factorise each of the following.

**a** $x^2 - 100$ **b** $p^2 - 25$ **c** $81 - y^2$ **d** $36 - x^2$
**e** $x^2 - 49$ **f** $1 - a^2$ **g** $y^2 - 144$ **h** $x^2 - 64$
**i** $4a^2 - 9$ **j** $16y^2 - 49$ **k** $k^2 - 36m^2$ **l** $4p^2 - 81q^2$
**m** $100 - y^2$ **n** $25 - x^2$ **o** $9a^2 - 4$ **p** $49b^2 - 16$
**q** $100a^2 - 49$ **r** $25x^2 - 1$ **s** $16b^2 - 25$ **t** $16 - 9x^2$
**u** $25 - 49m^2$ **v** $64 - 81a^2$ **w** $9x^2 - 169$ **x** $121x^2 - 49$

**4** Calculate each of the following without using a calculator.

**a** $17^2 - 3^2$ **b** $63^2 - 37^2$ **c** $59^2 - 41^2$ **d** $179^2 - 21^2$

LINKS TO Example 19b

**5** Factorise each of the following.

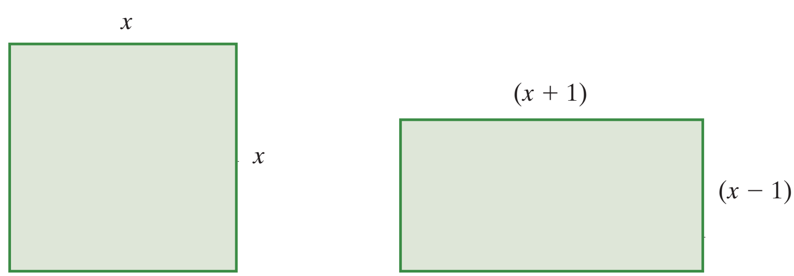
**a** $(a+2)^2-9$ **b** $(b+3)^2-25$ **c** $(c+8)^2-81$ **d** $(d-5)^2-36$
**e** $(e+7)^2-16$ **f** $(f-12)^2-100$ **g** $(g-4)^2-49$ **h** $(h+2)^2-64$
**i** $16-(i+5)^2$ **j** $4-(j+2)^2$ **k** $25-(k-3)^2$ **l** $9-(l-2)^2$

LINKS TO Example 20

**6** Factorise each of the following.

**a** $4x^2-36y^2$ **b** $3a^2-48b^2$ **c** $2x^2-98y^2$ **d** $7x^2-28$
**e** $-3x^2+75$ **f** $-2m^2+50n^2$ **g** $4x^2-400y^2$ **h** $6b^2-54$

**7** Ahmed has a square vegetable garden in his backyard as illustrated below.
In a new landscape design for the backyard, the square vegetable garden is changed into a rectangular one by adding 1 m to the length and subtracting 1 m from the width.




**a** Write an expression in terms of $x$ for the area of the original vegetable garden.
**b** Write an expression in terms of $x$ for the area of the new vegetable garden.
**c** Does the new garden have a larger area, a smaller area or the same area? Justify your answer.

LINKS TO Examples 21a, b

**8** Factorise each of the following over $R$.

**a** $x^2-5$ **b** $y^2-2$ **c** $7-a^2$
**d** $19-c^2$ **e** $p^2-12$ **f** $z^2-32$
**g** $3x^2-24$ **h** $5x^2-60$
**i** $25-5r^2$ **j** $49-7x^2$

$R$ means real numbers, so the factors will include irrational numbers.

LINKS TO Example 21c

**9** Factorise each of the following over $R$.

**a** $(x+4)^2-13$ **b** $(x+1)^2-6$
**c** $(x-2)^2-5$ **d** $(x-7)^2-10$
**e** $(x+3)^2-8$ **f** $(x-1)^2-18$
**g** $(x+1)^2-3$ **h** $(y+5)^2-7$
**i** $(p-3)^2-11$ **j** $(x+7)^2-8$
**k** $(x-1)^2-27$ **l** $15-(x+2)^2$
**m** $18-(k-2)^2$ **n** $(m-5)^2-5$

8 and 18 each have a square number as a factor.

$\sqrt{8}=2\sqrt{2}$ and $\sqrt{18}=3\sqrt{2}$

## exercise 4.5 challenge

**10** Factorise each of the following.

**a** $(4b+3)^2-9b^2$ **b** $(2x-1)^2-16x^2$

# 4.6 Factorising quadratic trinomials

An expression of the type $x^2 + 5x + 4$ is called a **quadratic trinomial**. The word 'quadratic' comes from the Latin *quad*, meaning 'square' or 'four', and refers to the '$x$ squared' term. A quadratic expression has a square as the highest power of the variable. The word 'trinomial' refers to the three terms.

We know from expanding two binomial factors that $(x + 3)(x + 4) = x^2 + 7x + 12$ and $(x - 4)(x - 2) = x^2 - 6x + 8$.

How do the 3 and 4 in $(x + 3)(x + 4)$ relate to the 7 and 12 in $x^2 + 7x + 12$?

How do the $-4$ and $-2$ in $(x - 4)(x - 2)$ relate to the $-6$ and 8 in $x^2 - 6x + 8$?

We observe that $3 + 4 = 7$ and $3 \times 4 = 12$. We also observe that $-4 - 2 = -6$ and $-4 \times (-2) = 8$.

**Factorising quadratic trinomials**

The distributive law gives the following expansion for $(x + a)(x + b)$.

$$\begin{aligned}(x + a)(x + b) &= x^2 + ax + bx + ab \\ &= x^2 + (a + b)x + ab\end{aligned}$$

$x^2 + (a + b)x + ab$ is a quadratic trinomial because it has an $x^2$ term, an $x$ term and a constant.

- The constant term is the *product* of the numbers in the brackets.
- The coefficient of $x$ is the *sum* of the numbers in the brackets.

Consider $x^2 - 2x - 15$. We need to find two numbers with a product of $-15$ and a sum of $-2$. The examples in the following table show the process of trial and error.

| Guess: pick numbers with a product of $-15$ | Check: Is the sum of the numbers $-2$? |
|---|---|
| 1 and $-15$ | $1 + -15 = -14$: no |
| 5 and $-3$ | $5 + -3 = 2$: no |
| 3 and $-5$ | $3 + -5 = -2$: yes |

Choose numbers that give the correct product and check whether they also give the correct sum.

Trial and error shows that the numbers are 3 and $-5$. Therefore

$x^2 - 2x - 15 = (x + 3)(x - 5)$

This can also be written as $(x - 5)(x + 3)$.

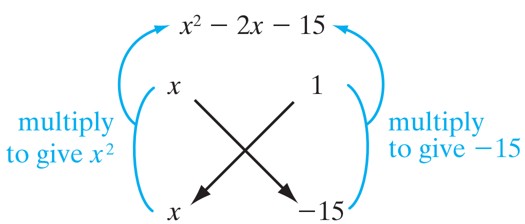
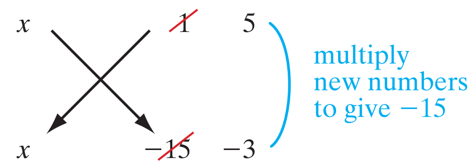
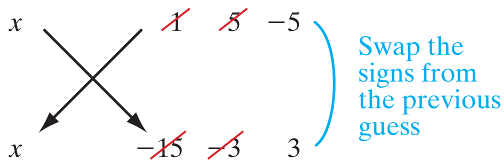
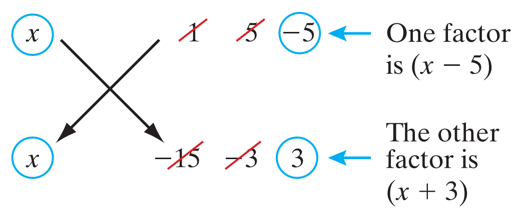
## The cross method

A visual representation of the trial-and-error process uses a cross as follows.

- Write the factors of the $x^2$ term on the left of the cross.
- Write your guesses for the numbers on the right.
- Cross-multiply to see if those numbers give the correct middle term.

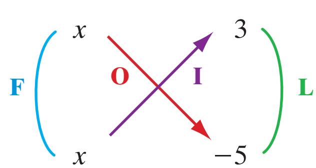
If the middle term is not correct, repeat the last two steps for another guess.

The following steps show this process for the guesses in the table above.

| **1st guess: 1 and −15** | **2nd guess: 5 and −3** |
|---|---|
| <br>$-15x^2 + 1x = -14x$<br>Cross-multiplying does *not* give the middle term. | <br>$-3x + 5x = 2x$<br>Cross-multiplying gives $2x$, not $-2x$. |
| **3rd guess: 3 and −5** | **The 'rows' in the cross diagram give the factors.** |
| <br>$3x - 5x = -2x$<br>Cross-multiplying gives $-2x$, which is the correct middle term. | <br>$x^2 - 2x - 15 = (x - 5)(x + 3)$<br>or $(x + 3)(x - 5)$ |

Notice that the cross method is using FOIL in reverse.

$$x^2 - 2x - 15$$




$$(x + 3)(x - 5)$$

## Example 22

Factorise each of the following.

**a** $x^2 + 11x + 24$ **b** $x^2 - 12x + 20$ **c** $x^2 + 5x - 24$

**Working** / **Reasoning**

**a** $x^2 + 11x + 24$

The constant term is positive, so both numbers will have the same sign.

The middle term is positive, so both numbers are positive.

$= (x + 3)(x + 8)$

$3 \times 8 = 24$ and $3 + 8 = 11$

**b** $x^2 - 12x + 20$

The constant term is positive, so both numbers will have the same sign, but the middle term is negative, so both numbers are negative.

$= (x - 2)(x - 10)$

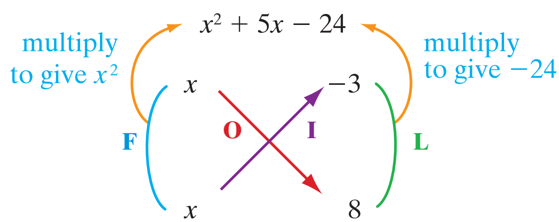
$-2 \times -10 = 20$ and $-2 + -10 = -12$

**c** $x^2 + 5x - 24$

$= (x - 3)(x + 8)$

The constant term is negative, so the numbers will have opposite signs.

The cross method is particularly helpful when the numbers have opposite signs.




$8x - 3x = 5x$

Cross-multiplying gives the correct middle term.

If the quadratic trinomial expression has a common factor, this should be taken out first.

If the coefficient of the $x^2$ term is $-1$, then $-1$ should be taken out as a factor.

## Example 23

Factorise the following expressions.

**a** $2x^2 + 2x - 24$ **b** $5x^2 - 15x + 10$ **c** $-x^2 - 10x + 24$

**Working** / **Reasoning**

**a** $2x^2 + 2x - 24$

$= 2(x^2 + x - 12)$

First take out the common factor of 2.

Within the brackets, the constant term is negative, so the numbers have opposite signs.

$= 2(x - 3)(x + 4)$

$-3 \times 4 = -12$ and $-3 + 4 = 1$

continued

## Example 23 continued

| Working | Reasoning |
|---|---|
| **b** $5x^2 - 15x + 10$ <br> $= 5(x^2 - 3x + 2)$ | First take out the common factor of 5. <br> The constant term is positive, so both numbers will have the same sign. <br> The middle term is negative, so both numbers are negative. |
| $= 5(x - 2)(x - 1)$ | $-2 \times -1 = 2$ and $-2 + -1 = -3$ |
| **c** $-x^2 - 10x + 24$ <br> $= -(x^2 + 10x - 24)$ <br> $= -(x + 12)(x - 2)$ | Care should be taken with the signs inside the bracket when the negative sign is taken out in front. <br> The factorised expression can be left as $-(x + 12)(x - 2)$ or it can be written as $(x + 12)(2 - x)$. |

## exercise 4.6

LINKS TO Example 22

**1** Factorise each of the following.

**a** $x^2 + 3x + 2$ **b** $x^2 + 4x + 3$ **c** $x^2 + 12x + 36$
**d** $x^2 + 14x + 40$ **e** $x^2 - 9x + 20$ **f** $x^2 + 4x - 12$
**g** $x^2 + 18x - 63$ **h** $x^2 - 13x + 36$ **i** $x^2 - 22x + 72$
**j** $x^2 + 12x - 45$ **k** $x^2 - 15x + 56$ **l** $x^2 + 10x - 56$
**m** $a^2 - 11a + 18$ **n** $a^2 - 5a - 24$ **o** $b^2 + 14b - 72$
**p** $d^2 - 11d - 60$ **q** $m^2 - 22m - 48$ **r** $y^2 + 18y + 45$

LINKS TO Example 23

**2** Factorise each of the following by first taking out the common factor.

**a** $5x^2 + 15x + 10$ **b** $3x^2 + 12x + 9$ **c** $5x^2 - 10x - 40$
**d** $5x^2 - 30x + 40$ **e** $4x^2 - 12x - 40$ **f** $2x^2 - 10x + 12$
**g** $3x^2 - 15x - 108$ **h** $5x^2 + 15x - 350$ **i** $2x^2 - 44x + 144$
**j** $2x^2 + 8x + 6$ **k** $3x^2 + 21x + 36$ **l** $8x^2 - 40x + 48$
**m** $-x^2 + 7x + 8$ **n** $-x^2 + 3x + 10$ **o** $-x^2 - 9x - 20$

## exercise 4.6 challenge

**3** Factorise these quadratic expressions.

**a** $x^2 - 4x - 221 = 0$ **b** $x^2 + 26x + 133 = 0$
**c** $x^2 + 5x - 104 = 0$ **d** $x^2 - 30x + 189 = 0$

# 4.7 Completing the square

Completing the square

Quadratic trinomials cannot always be factorised by the guess-and-check methods of inspection shown in the previous sections. In some cases, we may be able to use a method called **completing the square** to factorise a quadratic.

## How to complete the square

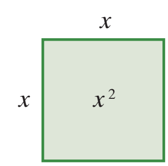
Suppose that we want to factorise $x^2 + 6x + 7$.

There are no factors of 7 that add to give 6, so we cannot factorise by inspection.

We use the first two terms of the quadratic to 'build' a perfect square, as follows.

### Step 1

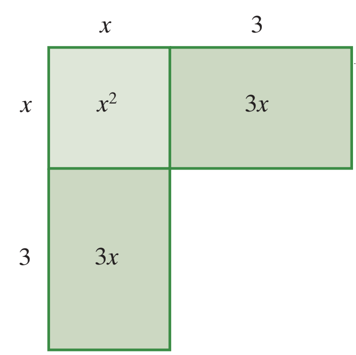
Start with a square of area $x^2$.




### Step 2

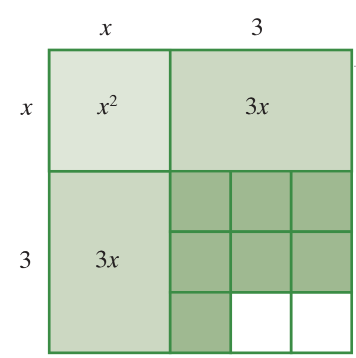
To show $6x$, add two rectangles, each of area $3x$.

This creates two sides of a square of side length $x + 3$, with a piece missing.




### Step 3

The 7 units can now be added.




### Step 4

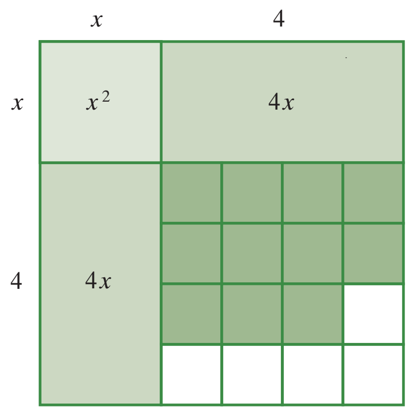
The diagram shows us that we need another 2 units to complete the square with area $(x + 3)^2$.

Since the area of $x^2 + 6x + 7$ is 2 units less than the area of the square, we can say
$x^2 + 6x + 7 = (x + 3)^2 - 2$

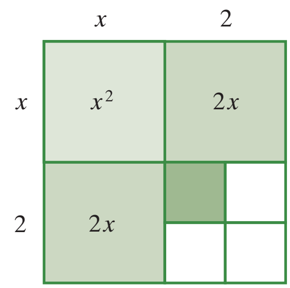
## Example 24

**a** Use the diagram to complete the square for each of the following and express in the form $(x + h)^2 - k$.

**i** $x^2 + 8x + 11$



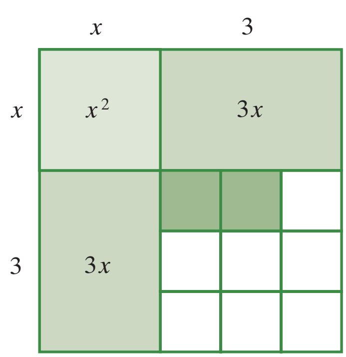

**ii** $x^2 + 4x + 1$




**b** Use a diagram to complete the square for each of the following.

**i** $x^2 + 6x + 2$ **ii** $x^2 + 10x + 6$

| Working | Reasoning |
|---|---|
| **a i** $x^2 + 8x + 11$<br>$= (x + 4)^2 - 5$ | $x^2 + 8x + 11$ is 5 less than $(x + 4)^2$ |
| **ii** $x^2 + 4x + 1$<br>$= (x + 2)^2 - 3$ | $x^2 + 4x + 1$ is 3 less than $(x + 2)^2$ |
| **b i** $x^2 + 6x + 2$<br>$= (x + 3)^2 - 7$ | Halve the coefficient of $6x$ and draw the diagram for $(x + 3)^2$<br>$x^2 + 6x + 2$ is 7 less than $(x + 3)^2$ |

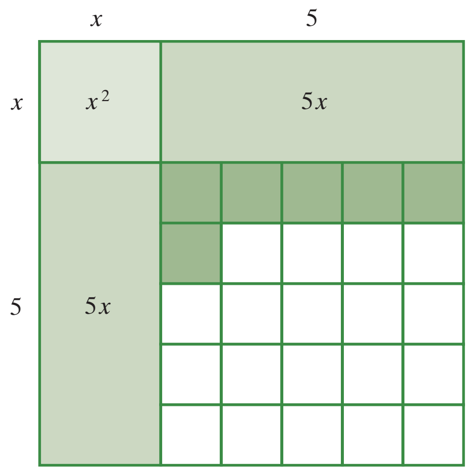



| Working | Reasoning |
|---|---|
| **ii** $x^2 + 10x + 6$<br>$= (x + 5)^2 - 19$ | Halve the coefficient of $10x$ and draw the diagram for $(x + 5)^2$<br>$x^2 + 10x + 6$ is 19 less than $(x + 5)^2$ |

continued

**Example 24** continued

**Working**




**Reasoning**

## Factorising by completing the square

We now look at the expressions we have obtained by completing the square to see how this helps us to factorise them.

We know from section 4.5 that the expression $(x + 4)^2 - 5$ is a difference of squares and that we can factorise it over $R$ as $(x + 4 - \sqrt{5})(x + 4 + \sqrt{5})$.

So, using the first expression in example 23a

$$\begin{aligned} x^2 + 8x + 11 &= (x + 4)^2 - 5 \\ &= (x + 4 - \sqrt{5})(x + 4 + \sqrt{5}). \end{aligned}$$

In a similar way we can express the other quadratic trinomial in example 23a.

$$\begin{aligned} x^2 + 4x + 1 &= (x + 2)^2 - 3 \\ &= (x + 2 - \sqrt{3})(x + 2 + \sqrt{3}) \end{aligned}$$

**Example 25**

Use the expressions obtained by completing the square to express each of these quadratic trinomials in factorised form.

**a** $x^2 + 6x + 2$

**b** $x^2 + 10x + 6$

**Working**

**a** $x^2 + 6x + 2$
$= (x + 3)^2 - 7$
$= (x + 3 - \sqrt{7})(x + 3 + \sqrt{7})$

**Reasoning**

$(x + 3)^2 - 7$ can be written as a difference of two squares:
$(x + 3)^2 - 7 = (x + 3)^2 - (\sqrt{7})^2$

continued

## Example 25 continued

**Working**

**b** $x^2 + 10x + 6$
$= (x + 5)^2 - 19$
$= (x + 5 - \sqrt{19})(x + 5 + \sqrt{19})$

**Reasoning**

$(x + 5)^2 - 19$ can be written as a difference of two squares:
$(x + 5)^2 - 19 = (x + 5)^2 - (\sqrt{19})^2$

## Example 26

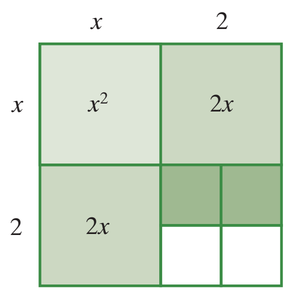
Use the method of completing the square to factorise the following expressions.

**a** $x^2 + 4x + 2$

**b** $x^2 + 10x + 13$

**Working**

**a** $x^2 + 4x + 2$
Construct a square of side length $x + 2$.



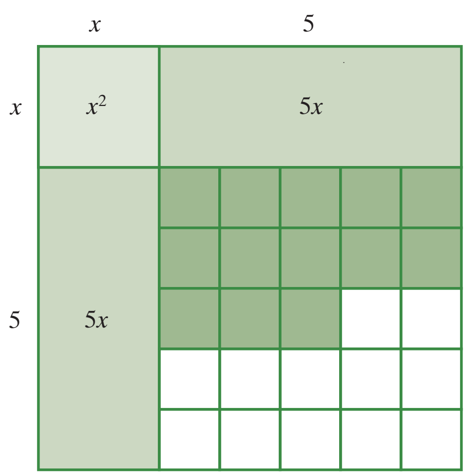

**Reasoning**

Two units are required to complete the square.
So $x^2 + 4x + 2$ is 2 units less than the square of side length $x + 2$.

So $x^2 + 4x + 2$
$= (x + 2)^2 - 2$
$= (x + 2)^2 - (\sqrt{2})^2$
$= (x + 2 + \sqrt{2})(x + 2 - \sqrt{2})$

This is a difference of squares.

**b** $x^2 + 10x + 13$
Construct a square of side length $x + 5$.




12 units are required to complete the square.
So $x^2 + 10x + 13$ is 12 units less than the square of side length $x + 5$.

continued

**Example 26** continued

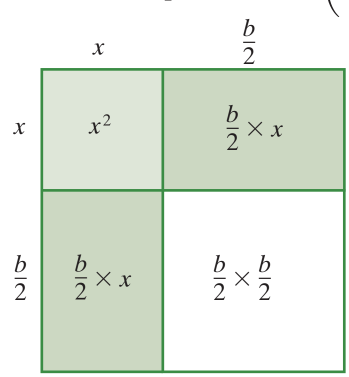
| **Working** | **Reasoning** |
|---|---|
| So $x^2 + 10x + 13$ | This is a difference of squares. |
| $= (x + 5)^2 - 12$ | $\sqrt{12}$ can be simplified: |
| $= (x + 5)^2 - (\sqrt{12})^2$ | $\sqrt{12} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$ |
| $= (x + 5 - \sqrt{12})(x + 5 + \sqrt{12})$ | |
| $= (x + 5 - 2\sqrt{3})(x + 5 + 2\sqrt{3})$ | |

# Completing the square algebraically

In order to avoid having to draw a diagram to complete the square, we can generalise from the patterns found in the numerical examples.

For a general expression of the form $x^2 + bx$, the square formed has side length $x + \frac{b}{2}$.

This results in a square of area $\left(x + \frac{b}{2}\right)^2$.




The region required to complete the square is $\left(\frac{b}{2}\right)^2 = \frac{b^2}{4}$

**Completing the square**

To complete the square for $x^2 + bx$, we add the square of half the coefficient of $x$.

That is, $x^2 + bx + \frac{b^2}{4}$ is always a perfect square.

$$x^2 + bx + \frac{b^2}{4} = \left(x + \frac{b}{2}\right)^2$$

## Example 27

Add the appropriate numbers to complete the square.

**a** $x^2 + 2x + \_\_ = (x + \_\_)^2$

**b** $x^2 + 14x + \_\_ = (x + \_\_)^2$

**c** $x^2 + 10x + \_\_ = (x + \_\_)^2$

| Working | Reasoning |
|---|---|
| **a** $x^2 + 2x + 1 = (x + 1)^2$ | Halve the coefficient of $x$ and square it.<br>$2 \div 2 = 1$<br>$1^2 = 1$<br>We must add 1 to make a perfect square. |
| **b** $x^2 + 14x + 49 = (x + 7)^2$ | $14 \div 2 = 7$<br>$7^2 = 49$<br>We must add 49 to make a perfect square. |
| **c** $x^2 + 10x + 25 = (x + 5)^2$ | $10 \div 2 = 5$<br>$5^2 = 25$<br>We must add 25 to make a perfect square. |

## Forming a difference of squares

Whenever we add a number to complete a square, we must also subtract that number to ensure that the overall expression is still equivalent to the original. As a result of that subtraction, we obtain a difference of squares. For example

$$\begin{aligned} x^2 + 6x &= \underbrace{x^2 + 6x + (3)^2} - (3)^2 \\ &= (x + 3)^2 - 9 \end{aligned}$$

To complete the square for $x^2 + 6x$, take half of 6 and square it.

Applying this method to the expression $x^2 + 6x + 1$, we obtain

$$\begin{aligned} x^2 + 6x + 1 &= \underbrace{x^2 + 6x + (3)^2} + 1 - (3)^2 \\ &= \underbrace{(x + 3)^2} + 1 - 9 \\ &= (x + 3)^2 - 8 \end{aligned}$$

We can then factorise this difference of squares over $R$.

$x^2 + 6x + 1 = (x + 3 + \sqrt{8})(x + 3 - \sqrt{8})$

The method of completing the square does not always give a *difference* of two squares. Sometimes the result of completing the square is a *sum* of two squares. A sum of squares cannot be factorised.

## Example 28

For which of the following do you end up with a *sum* of squares rather than a *difference* of squares?

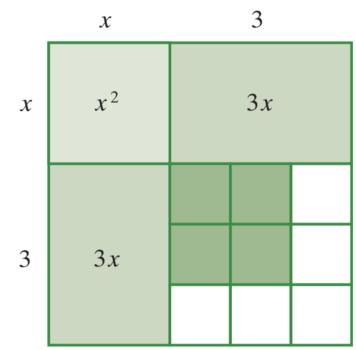
**a** $x^2 + 6x + 4$ **b** $x^2 + 4x + 5$ **c** $x^2 - 8x + 20$ **d** $x^2 + 2x + 3$

**Working**

**a** $x^2 + 6x + 4$
$= x^2 + 6x + 9 - 5$
$= (x + 3)^2 - 5$
This is a difference of squares.

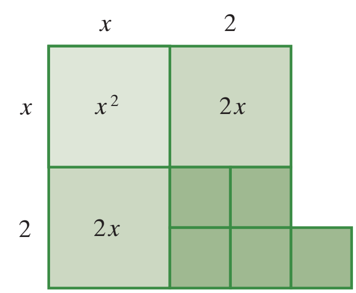
**Reasoning**

We need to add 5 to complete the square so we have to subtract 5 to keep the expression the same.




**b** $x^2 + 4x + 5$
$= x^2 + 4x + 4 + 1$
$= (x + 2)^2 + 1$
This is a sum of squares.

Instead of having to complete the square, the square is already complete with an extra 1.
$x^2 + 4x + 5$ cannot be factorised.




**c** $x^2 - 8x + 20$
$= x^2 - 8x + 16 + 4$
$= (x - 4)^2 + 4$
This is a sum of squares.

The square is already complete with an extra 4.
$x^2 - 8x + 20$ cannot be factorised.

**d** $x^2 + 2x + 3$
$= x^2 + 2x + 1 + 2$
$= (x + 1)^2 + 2$
This is a sum of squares.

The square is already complete with an extra 2.
$x^2 + 2x + 3$ cannot be factorised.

## Example 29

Use the method of completing the square to express each of the following expressions as a difference of squares. Hence factorise each expression.

**a** $x^2 + 8x - 5$ **b** $x^2 - 4x + 2$ **c** $x^2 + 14x + 17$

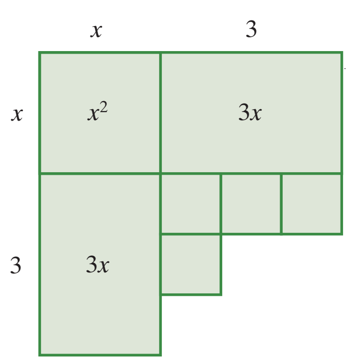
| Working | Reasoning |
|---|---|
| **a** $x^2 + 8x - 5$ | |
| $= \underbrace{x^2 + 8x + (4)^2}_{} \underbrace{- 5 - 16}_{}$ | To complete the square, add the square of half of 8, that is, $4^2$. Subtract the same amount, that is, 16. |
| $= (x + 4)^2 - 21$ | |
| $= (x + 4)^2 - (\sqrt{21})^2$ | This gives a difference of squares. |
| $= (x + 4 + \sqrt{21})(x + 4 - \sqrt{21})$ | Use $a^2 - b^2 = (a - b)(a + b)$, where $a$ is $x + 4$ and $b$ is $\sqrt{21}$. |
| **b** $x^2 - 4x + 2$ | Add the square of half of 4, that is, $2^2$. |
| $= \underbrace{x^2 - 4x + (2)^2}_{} \underbrace{+ 2 - 4}_{}$ | Subtract the same amount, that is, 4. |
| $= (x - 2)^2 - 2$ | This gives a difference of squares. |
| $= (x - 2)^2 - (\sqrt{2})^2$ | |
| $= (x - 2 + \sqrt{2})(x - 2 - \sqrt{2})$ | |
| **c** $x^2 + 14x + 17$ | Add the square of half of 14, that is, $7^2$. |
| $= \underbrace{x^2 + 14x + (7)^2}_{} \underbrace{+ 17 - 49}_{}$ | Subtract the same amount, that is, 49. |
| $= (x + 7)^2 - 32$ | This gives a difference of squares. |
| $= (x + 7)^2 - (\sqrt{32})^2$ | |
| $= (x + 7 + \sqrt{32})(x + 7 - \sqrt{32})$ | $\sqrt{32}$ can be simplified: |
| $= (x + 7 + 4\sqrt{2})(x + 7 - 4\sqrt{2})$ | $\sqrt{32} = \sqrt{16} \times \sqrt{2} = 4\sqrt{2}$ |

## exercise 4.7

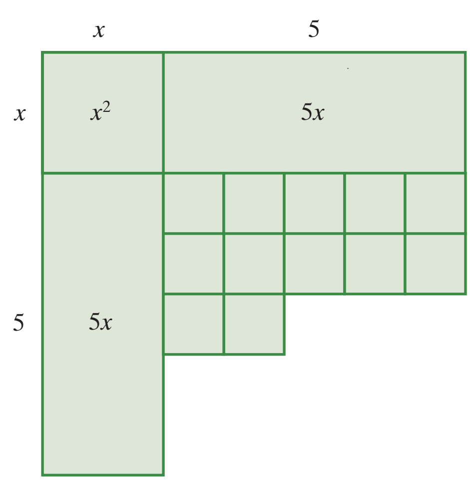
LINKS TO Example 24

**1** Use the diagrams to complete the square for each of these quadratic expressions and then, write each expression in the form $(x + h)^2 - k$.

**a** $x^2 + 6x + 4$




**b** $x^2 + 10x + 12$



LINKS TO Example 24

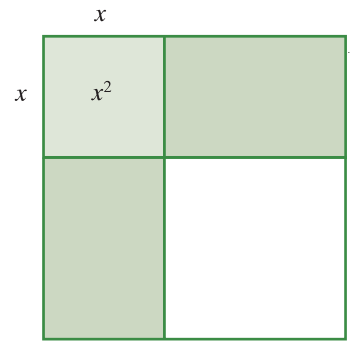
**2** Use a diagram to complete the square for each of the following. Hence write each expression in the form $(x + h)^2 - k$.

**a** $x^2 + 12x$

**b** $x^2 + 16x$

**c** $x^2 + 20x$

**d** $x^2 + 8x$




LINKS TO Example 24

**3** Draw diagrams to help you complete the square for each of the following, and then write each expression in the form $(x - h)^2 + k$.

**a** $x^2 + 4x + 3$ **b** $x^2 + 8x + 7$ **c** $x^2 + 10x + 15$

**d** $x^2 + 6x + 2$ **e** $x^2 + 12x + 20$ **f** $x^2 + 2x$

LINKS TO Examples 25, 26

**4** Factorise each of the following.

**a** $(x + 5)^2 - 6$ **b** $(x - 8)^2 - 11$ **c** $(x + 1)^2 - 20$ **d** $(x - 12)^2 - 48$

**e** $(x + 3)^2 - 3$ **f** $(x - 2)^2 - 5$ **g** $(x + 4)^2 - 18$ **h** $(x - 7)^2 - 32$

**i** $(x + 2)^2 - 24$ **j** $(x + 1)^2 - 40$ **k** $(x - 1)^2 - 50$ **l** $(x + 6)^2 - 27$

LINKS TO Example 27

**5** Add the appropriate number to each of the following to complete the square.

**a** $x^2 + 6x + \underline{\phantom{0}} = (x + \underline{\phantom{0}})^2$ **b** $x^2 + 16x + \underline{\phantom{0}} = (x + \underline{\phantom{0}})^2$

**c** $x^2 + 18x + \underline{\phantom{0}} = (x + \underline{\phantom{0}})^2$ **d** $x^2 + 22x + \underline{\phantom{0}} = (x + \underline{\phantom{0}})^2$

LINKS TO Example 28

**6** How would you write the following expressions in the form $(x + h)^2 + k$?

**a** $x^2 + 6x + 12$ **b** $x^2 + 2x + 5$

LINKS TO Example 29

**7** Use the method of completing the square to factorise each of the following.

**a** $x^2 + 12x + 30$ **b** $x^2 - 10x + 20$ **c** $x^2 + 14x + 32$ **d** $x^2 + 4x - 1$

**e** $x^2 + 22x + 100$ **f** $x^2 - 6x - 2$ **g** $x^2 + 18x + 41$ **h** $x^2 + 10x - 4$

**i** $x^2 + 8x - 4$ **j** $x^2 - 16x + 40$ **k** $x^2 + 24x + 100$ **l** $x^2 + 8x + 4$

**m** $x^2 + 8x - 13$ **n** $x^2 - 2x - 5$ **o** $x^2 + 4x - 7$ **p** $x^2 + 14x - 3$

## exercise 4.6 challenge

**8** Use the method of completing the square to factorise each of the following.

**a** $x^2 + 5x + 1$ **b** $x^2 + 9x + 6$

# 4.8 Further factorisation of quadratic expressions

## Non-monic quadratic trinomials

In the quadratic trinomials factorised so far, the coefficient of the $x^2$ term was 1 or there was a common factor leaving a coefficient of 1 in the quadratic expression to be factorised. Quadratic trinomials where the coefficient of $x^2$ is 1 are called **monic quadratic trinomials**. We now look at the factorisation of **non-monic quadratic equations**.

It is more difficult to factorise a quadratic expression where there is no common factor and the coefficient of $x^2$ is not $+1$ or $-1$.

The cross method that was introduced in section 4.6 is useful in working out the required terms of the factors.

## Differences of squares: some further examples

A difference of squares may be a difference of two squared binomial expressions. It is often easier to substitute $a$ and $b$ for the binomial expressions, factorise $a^2 - b^2$, then substitute the binomial expressions back into $(a - b)(a + b)$ and finally simplify the expression in each bracket.

### Example 30

Factorise each of the following.

**a** $(x + y)^2 - (2x - y)^2$ **b** $4(a + 1)^2 - 16(2a - 3)^2$ **c** $x^2 - y^2 + x + y$

| Working | Reasoning |
| --- | --- |
| **a** $(x + y)^2 - (2x - y)^2$<br>$= [x + y + (2x - y)][x + y - (2x - y)]$<br>$= (3x)(x + y - 2x + y)$<br>$= 3x(2y - x)$<br>Be careful with $-$ signs! | Use $a^2 - b^2 = (a - b)(a + b)$ where $a$ is $x + y$ and $b$ is $2x - y$.<br>It is useful to use two types of brackets to avoid confusion. |
| **b** $4(a + 1)^2 - 16(2a - 3)^2$ | Take out the common factor of 4. |
| $= 4[(a + 1)^2 - 4(2a - 3)^2]$<br>$= 4[(a + 1)^2 - (2(2a - 3))^2]$<br>$= 4[(a + 1 + 2(2a - 3)][a + 1 - 2(2a - 3)]$ | Use $a^2 - b^2 = (a - b)(a + b)$ to factorise the expression inside the square brackets. |
| $= 4(a + 1 + 4a - 6)(a + 1 - 4a + 6)$ | Simplify the brackets. |
| $= 4(5a - 5)(7 - 3a)$<br>$= 20(a - 1)(7 - 3a)$ | Take out the common factor of 5 from the first bracket. |

continued

## Example 30 continued

**Working**

**c** $x^2 - y^2 + x + y$

$= (x - y)(x + y) + 1(x + y)$

$= (x + y)(x - y + 1)$

**Reasoning**

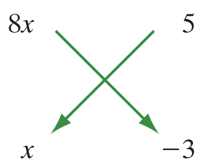
Group 'two and two', and use the difference of squares rule on the first pair of terms.

Take out the common factor of $x + y$.

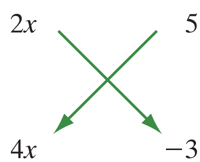
## Example 31

Which of these cross diagrams will give the factors of $8x^2 - 2x - 15$?

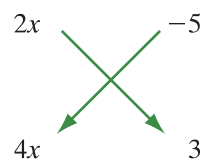
**A**



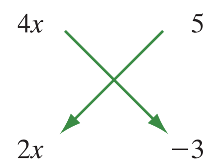

**B**




**C**




**D**




**Working**

**D** $8x^2 - 2x - 15 = (4x + 5)(2x - 3)$

**Reasoning**

**A** gives $(8x + 5)(x - 3) = 8x^2 - 19x - 15$

**B** gives $(2x + 5)(4x - 3) = 8x^2 + 12x - 15$

**C** gives $(2x - 5)(4x + 3) = 8x^2 - 16x - 15$

So all correctly give $8x^2$ and $-15$, but only D gives $-2x$ as the middle term.

## Example 32

Factorise the following expressions.

**a** $3x^2 + 2x - 8$ **b** $6x^2 - 13x + 5$ **c** $9x^2 - 21x + 10$

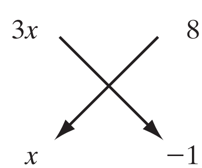
**Working**

**a** $3x^2 + 2x - 8$

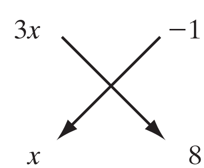
**Reasoning**

The only suitable factors of $3x^2$ are $3x$ and $x$, so these are the 'firsts'. The constant term is negative, so the 'lasts' will have different signs. Test pairs of numbers with a product of $-8$.

If the $x$-term is correct except for the sign, simply swap the signs of the 'lasts'.




$-3x + 8x = 5x$




$24x - 1x = 23x$

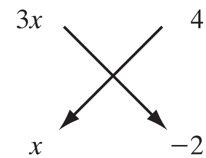
continued

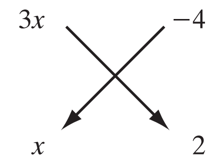
## Example 32 continued

**Working**

$= (3x - 4)(x + 2)$

**Reasoning**



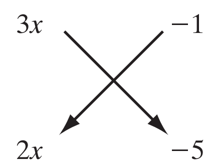



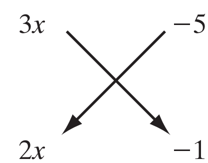

$-6x + 4x = -2x$ $\quad$ $6x - 4x = 2x$

**b** $6x^2 - 13x + 5$

The constant term is positive, so the 'lasts' will have the same sign.
The middle term is negative, so the 'lasts' must be $-1$ and $-5$.
The 'firsts' could be $6x$ and $x$ or $3x$ and $2x$. Use the cross method to check.

$= (3x - 5)(2x - 1)$



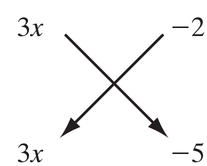




$-15x - 2x = -17x$ $\quad$ $-3x - 10x = -13x$

**c** $9x^2 - 21x + 10$

$= (3x - 2)(3x - 5)$

Both last terms will be negative.
Using the cross method to guess and check eventually gives




$-15x - 6x = -21x$

Sometimes it is necessary to take out a common factor first.

## Example 33

Factorise the following expressions.

$-27x^2 + 90xy - 75y^2$

**Working**

$-27x^2 + 90xy - 75y^2$

$= -3(9x^2 - 30xy + 25y^2)$

$= -3[(3x)^2 - 2 \times 3x \times 5y + (5y)^2]$

$= -3(3x - 5y)^2$

**Reasoning**

Take out a common factor of $-3$.

$9x^2 - 30xy + 25y^2$ can be written in the form $a^2 - 2ab + b^2$, where $a = 3x$ and $b = -5y$.

$a^2 - 2ab + b^2 = (a - b)^2$

## Example 34

Factorise each of the following.

**a** $4x^2 + 20xy + 25y^2$

**b** $a^2 - \frac{2}{3}a + \frac{1}{9}$

**Working**

**a** $4x^2 + 20xy + 25y^2$
$= (2x)^2 + 2 \times 2x \times 5y + (5y)^2$
$= (2x + 5y)^2$

**Reasoning**

$4x^2 + 20xy + 25y^2$ is a perfect square of the form $a^2 + 2ab + b^2 = (a + b)^2$, where $a$ is $2x$ and $b$ is $5y$.

**Working**

**b** $a^2 - \frac{2}{3}a + \frac{1}{9}$
$= a^2 - 2 \times a \times \frac{1}{3} + \left(\frac{1}{3}\right)^2$
$= \left(a - \frac{1}{3}\right)^2$

**Reasoning**

$a^2 - \frac{2}{3}a + \frac{1}{9}$ is a perfect square of the form $a^2 - 2ab + b^2 = (a - b)^2$, where $b$ is $\frac{1}{3}$.

## Factorising quadratics by substitution

Some trinomial expressions can be factorised by temporarily substituting a different pronumeral (such as $y$) for an algebraic expression (such as $x^2$). The following example illustrates this idea.

## Example 35

Factorise each of the following expressions.

**a** $36x^4 - 25x^2 + 4$

**b** $(x^2 + 4x)^2 - 2(x^2 + 4x) - 15$

**Working**

**a** $36x^4 - 25x^2 + 4$
$= 36y^2 - 25y + 4$, where $y = x^2$
$= (4y - 1)(9y - 4)$

**Reasoning**

If $x^2$ is replaced by a single pronumeral, $y$, the expression becomes a quadratic trinomial.
Trial and error gives

$$\begin{array}{cc} 4y & -1 \\ 9y & -4 \end{array}$$

$-16y - 9y = -25y$

**Working**

$= (4x^2 - 1)(9x^2 - 4)$
$= (2x - 1)(2x + 1)(3x - 2)(3x + 2)$

**Reasoning**

Now substitute $x^2$ for $y$.
Each difference of squares can be factorised.

continued

**Example 35** continued

**Working**

**b** $(x^2 + 4x)^2 - 2(x^2 + 4x) - 15$
$= b^2 - 2b - 15$, where $b = x^2 + 4x$

$= (b + 3)(b - 5)$

$= (x^2 + 4x + 3)(x^2 + 4x - 5)$
$= (x + 3)(x + 1)(x + 5)(x - 1)$

**Reasoning**

If $x^2 + 4x$ is replaced by a single pronumeral, $b$, this becomes a quadratic trinomial.
Trial and error gives

$$\begin{array}{ccc} b & & 3 \\ & \times & \\ b & & -5 \end{array}$$

$-5b + 3b = -2b$

Now substitute $x^2 + 4x$ for $b$.
Each of the quadratic trinomials can be factorised.

# Completing the square with fractions

If the coefficient of $x$ is odd, completing the square will involve fractions.

**Example 36**

Use the method of completing the square to express each of the following expressions as a difference of squares. Hence factorise each expression.

**a** $x^2 + 5x + 3$ **b** $x^2 - 7x - 3$

**Working**

**a** $x^2 + 5x + 3$

$$= \underbrace{x^2 + 5x + \left(\frac{5}{2}\right)^2}\ \underbrace{+\ 3 - \frac{25}{4}}$$

$$= \left(x + \frac{5}{2}\right)^2 + \frac{12 - 25}{4}$$

$$= \left(x + \frac{5}{2}\right)^2 - \frac{13}{4}$$

$$= \left(x + \frac{5}{2}\right)^2 - \frac{\sqrt{13}}{2}$$

$$= \left(x + \frac{5}{2} + \frac{\sqrt{13}}{2}\right)\left(x + \frac{5}{2} - \frac{\sqrt{13}}{2}\right)$$

**Reasoning**

Add the square of half of 5, or $\left(\frac{5}{2}\right)^2$.

Subtract that amount, that is, $\frac{25}{4}$.

Write 3 as a fraction with the same denominator as $\frac{25}{4}$. Add the fractions.

This gives a difference of squares $a^2 - b^2$, where $a$ is $x + \frac{5}{2}$ and $b$ is

$$\sqrt{\frac{13}{4}} = \frac{\sqrt{13}}{\sqrt{4}} = \frac{\sqrt{13}}{2}$$

continued

## Example 36 continued

**Working**

**b** $x^2 - 7x - 3$

$$= \underbrace{x^2 - 7x^2 + \left(\frac{7}{2}\right)^2}\ \underbrace{-\,3 - \frac{49}{4}}$$

$$= \left(x - \frac{7}{2}\right)^2 - \frac{12 + 49}{4}$$

$$= \left(x - \frac{7}{2}\right)^2 - \frac{61}{4}$$

$$= \left(x - \frac{7}{2}\right)^2 - \frac{\sqrt{61}}{2}$$

$$= \left(x - \frac{7}{2} + \frac{\sqrt{61}}{2}\right)\left(x - \frac{7}{2} - \frac{\sqrt{61}}{2}\right)$$

**Reasoning**

Add the square of half of 7, or $\left(\frac{7}{2}\right)^2$.

Subtract that amount, that is, $\frac{49}{4}$.

Write 3 as a fraction with the same denominator as $\frac{49}{4}$. Add the fractions.

This gives a difference of squares.

## exercise 4.8

LINKS TO Example 30

**1** Factorise each of the following.

**c** $(2x + 1)^2 - (x - 1)^2$ **d** $(a + b)^2 - (a - 2b)^2$ **e** $4(p + q)^2 - (2p + q)^2$

**f** $9(x - 1)^2 - 4(3 - x)^2$ **g** $x^2 - y^2 + 2x - 2y$

LINKS TO Example 31

**2** Factorise each of the following.

**a** $2x^2 + 7x + 3$ **b** $3x^2 + 10x + 3$ **c** $2x^2 + 7x + 6$

**d** $3x^2 + 5x + 2$ **e** $2x^2 + 11x + 5$ **f** $7x^2 + 10x + 3$

**g** $3x^2 + 22x + 7$ **h** $2x^2 - 5x - 3$ **i** $5x^2 + 7x + 2$

**j** $3x^2 + x - 2$ **k** $2x^2 - 5x + 3$ **l** $3x^2 - 2x - 5$

LINKS TO Example 32

**3** Factorise each of the following.

**a** $12x^2 + 7x + 1$ **b** $8x^2 + 18x - 5$ **c** $8x^2 - 2x - 15$

**d** $15x^2 + 26x + 8$ **e** $21x^2 - 20x + 4$ **f** $35x^2 - 9x - 2$

**g** $15 + 2x - 8x^2$ **h** $4 - 3x - x^2$ **i** $6 - 7x - 3x^2$

**j** $10x^2 - x - 2$ **k** $4x^2 - 11x + 6$ **l** $3x^2 - 7x - 6$

**m** $5x^2 + 11x + 2$ **n** $5x^2 - 22x + 21$ **o** $3x^2 - x - 30$

**p** $7x^2 - 2x - 5$ **q** $4x^2 - x - 18$ **r** $10x^2 + x - 2$

**s** $8x^2 + 38x + 35$ **t** $6x^2 - 13x + 2$ **u** $7x^2 - 11x - 6$

LINKS TO Example 33

**4** Factorise each of the following by first taking out the common factor.

**a** $8x^2 + 12x - 36$ **b** $30x^2 + 55x - 35$ **c** $6x^2 + 9x - 27$

**d** $12x^2 + 24x + 9$ **e** $50x^2 + 15x - 5$ **f** $4 - 6x - 10x^2$

**g** $6x^2 - 9x - 15$ **h** $12x^2 - 68x + 40$ **i** $30x^2 - 5x - 10$

LINKS TO Example 34

**5** Factorise these perfect squares.

**a** $4x^2 + 12xy + 9y^2$ **b** $4a^2 + 28ab + 49b^2$ **c** $9x^2 - 30xy + 25y^2$

**d** $16x^2 - 24xy + 9y^2$ **e** $25m^2 - 70mn + 49n^2$ **f** $49x^2 - 112xy + 64y^2$

**g** $x^2 - \frac{1}{4}x + \frac{1}{64}$ **h** $a^2 - \frac{1}{2}a + \frac{1}{16}$ **i** $x^2 + \frac{4}{5}x + \frac{4}{25}$

**6** Factorise each of the following.

**a** $x^3 + 5x^2 + 4x$ **b** $3x^3 + 4x^2 - 4x$ **c** $8x^3 + 18x^2 - 5x$

**d** $6x^4 + 5x^3 + x^2$ **e** $10x^4 - x^3 - 2x^2$ **f** $3x^4 - 15x^3 - 72x^2$

LINKS TO Example 35

**7** Use substitution to fully factorise each of the following.

**a** $x^4 - 5x^2 + 4$ **b** $x^4 - 29x^2 + 100$

**c** $16x^4 - 40x^2 + 9$ **d** $2(x + 3)^2 - (x + 3) - 6$

**e** $3(x - 1)^2 + 8(x - 1) + 4$ **f** $3(x + 1)^2 + 16(x + 1) + 5$

**g** $(x^2 + 3x)^2 - 8(x^2 + 3x) - 20$ **h** $(x^2 - x)^2 - 8(x^2 - x) + 12$

LINKS TO Example 36

**8** Use the method of completing the square to factorise each of the following.

**a** $x^2 + 11x + 20$ **b** $x^2 - 9x + 12$ **c** $x^2 + 15x + 32$

**d** $x^2 + 3x - 1$ **e** $x^2 + 13x - 50$ **f** $x^2 - 7x - 2$

**9** Fill the gaps to factorise each of the following.

**a** $2x^2 + 8x + 2$
$= 2[x^2 + \_\_\_ + \_\_\_]$
$= 2[x^2 + \_\_\_ + (\_\_\_)^2 + \_\_\_ - (\_\_\_)^2]$
$= 2[(x + \_\_\_)^2 - \_\_\_]$
$= 2(x + \_\_\_ + \_\_\_)(x + \_\_\_ - \_\_\_)$

**b** $3x^2 + 12x + 6$
$= 3[x^2 + \_\_\_ + \_\_\_]$
$= 3[x^2 + \_\_\_ + (\_\_\_)^2 + \_\_\_ - (\_\_\_)^2]$
$= 3[(x + \_\_\_)^2 - \_\_\_]$
$= 3(x + \_\_\_ + \_\_\_)(x + \_\_\_ - \_\_\_)$

**c** $2x^2 + 10x + 4$
$= 2[x^2 + \_\_\_ + \_\_\_]$
$= 2[x^2 + \_\_\_ + (\_\_\_)^2 + \_\_\_ - (\_\_\_)^2]$
$= 2[(x + \_\_\_)^2 - \_\_\_]$
$= 2(x + \_\_\_ + \_\_\_)(x + \_\_\_ - \_\_\_)$

**d** $5x^2 + 10x + 2$
$= 5[x^2 + \_\_\_ + \_\_\_]$
$= 5[x^2 + \_\_\_ + (\_\_\_)^2 + \_\_\_ - (\_\_\_)^2]$
$= 5[(x + \_\_\_)^2 - \_\_\_]$
$= 5(x + \_\_\_ + \_\_\_)(x + \_\_\_ - \_\_\_)$

**10** Take out a common factor and use the method of completing the square to factorise each of the following.

**a** $2x^2 - 16x + 4$ **b** $6x^2 + 24x - 12$ **c** $4x^2 + 32x + 44$

**d** $5x^2 + 30x + 30$ **e** $3x^2 + 9x + 3$ **f** $2x^2 + 4x - 10$

## exercise 4.8 challenge

**11** A certain circle has area $A = \pi(4x^2 + 20x + 25)$. Find an expression for the radius of this circle.

# Analysis task

BLM
Pascal's triangle

## Pascal's triangle and binomial expansions

Pascal's triangle is named after the French mathematician Blaise Pascal (1623–1662). Pascal was the first European to make an extensive study of the patterns in this special triangle, which was discovered by the Chinese about 500 years before Pascal was born.

The following steps are used to create Pascal's triangle.

- Write the number 1 to create the apex (top) of the triangle. This is row 0.
- Write the number 1 at either end of the next row. This is row 1.
- Write the number 1 at either end of row 2. To find the number in between, add the two numbers diagonally above.

| | | | | | |
|---|---|---|---|---|---|
| Row 0 | | | 1 | | |
| Row 1 | | 1 | | 1 | |
| Row 2 | 1 | | 2 | | 1 |

- To generate each new row in Pascal's triangle, write the number 1 at either end of the row. All numbers in between are found by adding the two numbers diagonally above.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Row 0 | | | | | | 1 | | | | | |
| Row 1 | | | | | 1 | | 1 | | | | |
| Row 2 | | | | 1 | | 2 | | 1 | | | |
| Row 3 | | | 1 | | 3 | | 3 | | 1 | | |
| Row 4 | | 1 | | 4 | | 6 | | 4 | | 1 | |
| Row 5 | 1 | | 5 | | 10 | | 10 | | 5 | | 1 |

**a** Copy Pascal's triangle above, and write the next three rows.

**b** Expand $(x + a)^2$. How does the expanded form relate to row 2 of Pascal's triangle? Hint: remember that the coefficient of $x^2 = 1$.

**c** Expand $(x + a)^3$ by using $(x + a)^3 = (x + a)(x + a)^2$, that is, multiply your previous answer by $(x + a)$. How does the expanded form relate to row 3 of Pascal's triangle?

**d** The expansion of $(x + a)^4$ is $x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4$. How does the expanded form relate to row 4 of Pascal's triangle?

**e** Look from one term to the next in $x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4$, starting with $x^4$.

**i** What pattern can you see in the powers of $x$?

**ii** What pattern can you see in the powers of $a$?

**f** Use Pascal's triangle to complete the following expansion.

$(x + a)^5 = x^5 + \underline{\quad}\,x^4a + \underline{\quad}\,x^3a^2 + \underline{\quad}\,x^2a^3 \underline{\quad}\,xa^4 + a^5$

**g** Use Pascal's triangle to help you expand $(x + a)^6$ and$(x + a)^7$.

### Challenge

**h** Expand $(x - a)^2$ and $(x - a)^3$. How are the results different from those for $(x + a)^2$ and $(x + a)^3$?

**i** Use Pascal's triangle to help you expand $(x - a)^4$ and $(x - a)^5$.

**j** Use Pascal's triangle to help you expand each of the following.

**i** $(x + 2)^4$ **ii** $(x - 3)^5$ **iii** $(2x + 1)^3$ **iv** $(2x + 3y)^4$

# Review Algebra

## Summary

### Algebraic expressions: substitution, expansion and common factors

- Replacing the pronumerals in an expression with particular numbers is called substitution. Filling in a table of values is a process of repeated substitution.
- The process of going from the factorised form of an expression to the expanded form is called expansion.
- In general, we can say that $a(b + c) = ab + ac$. This is the distributive law.
- When expanding two brackets, $(a + b)(c + d) = ac + ad + bc + bd$. The word FOIL reminds us to multiply Firsts, Outers, Inners and Lasts.

$$(a + b)(c + d)$$

- The process of going from the expanded form to the factorised form is called factorisation.
- The highest common factor of the terms in the expression is taken out in front of brackets.

### Factorisation involving binomial factors

- A binomial factor can be taken out in front of a bracket like any other common factor.
- To factorise by grouping two and two, group the terms in pairs so that each pair of terms has a common factor, and then take a binomial common factor out in front of a bracket.

### Perfect squares and differences of squares

Expansion

$$(a + b)^2 = a^2 + 2ab + b^2$$

Factorisation

Expansion

$$(a - b)^2 = a^2 - 2ab + b^2$$

Factorisation

Expansion

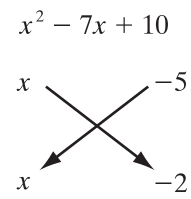
$$(a + b)(a - b) = a^2 - b^2$$

Factorisation

- Not all expressions can be factorised using only rational numbers. When factorisation requires the use of irrational square roots, such as $x^2 - 3 = (x + \sqrt{3})(x - \sqrt{3})$, this is referred to as factorisation over $R$, the field of real numbers.
- Expressions that are a *sum* of two squares, for example, $x^2 + 9$, cannot be factorised.

## Factorising quadratic trinomials

- To factorise a quadratic trinomial of the form $x^2 + bx + c$, look for two numbers that have
  - a *product* equal to $c$ (the constant term) and
  - a *sum* equal to $b$ (the coefficient of $x$).
- The cross-method is useful in factorising many quadratic trinomials. For non-monic quadratic trinomials check first to see if there is a common factor.




## Completing the square

- The method of completing the square is used to factorise quadratic trinomials that cannot be factorised using methods of inspection.

# Visual map

Using the following terms (and others if you wish), construct a visual map that illustrates your understanding of the key ideas covered in this chapter.

| | | |
|---|---|---|
| binomial expansion | expansion | monic |
| coefficient | expression | non-monic |
| common factor | factor | perfect square |
| complete the square | factorisation | power |
| difference of two squares | factorisation over *Q* | quadratic trinomial |
| equivalent expressions | factorisation over *R* | |
| expanded form | factorised form | |

# Revision

## Multiple-choice questions

**1** When factorised, $4p^2 - 16q^2$ is equivalent to

**A** $4(p - 2q)^2$ **B** $16(p - 4q)^2$ **C** $2(p - 4q)(p + 4q)$
**D** $4(p - 4q)(p + 4q)$ **E** $4(p - 2q)(p + 2q)$

**2** Which of the following expressions would be factorised by grouping 'two and two'?

**A** $x^2 - p^2 + 12p - 36$ **B** $x^2 - 7x - 10$ **C** $2x^2 - 6x - yx + 3y$
**D** $(w - 5)^2 - 25(w + 3)^2$ **E** $(b + 5) - (b + 3)(b + 5)$

**3** The factorised form of the expression $x^2 - 4x + 2$ is

**A** $(x + 4 + \sqrt{2})(x + 4 - \sqrt{2})$ **B** $(x - 3 + \sqrt{6})(x - 3 + \sqrt{6})$
**C** $(x + 2 + \sqrt{3})(x + 2 - \sqrt{3})$ **D** $(x - 2 + \sqrt{2})(x - 2 - \sqrt{2})$
**E** $(x - 2)^2$

**4** $(x + y)^2 - (x^2 + y^2)$ is equivalent to

**A** 0 **B** $xy$ **C** $2y^2$ **D** $4xy + 2y^2$ **E** $2xy$

**5** Which of the following is an incorrect simplification?

**A** $m^3 + m^3 = 2m^3$ **B** $3m^2 \times 27m^6 = 3^4m^8$ **C** $36m^{10} \div 18m^5 = 2m^2$
**D** $(m^3)^2 \times (m^2)^3 = m^{12}$ **E** $(3m)^2 \times 2n^4 = 18m^2n^4$

**6** The expression $x(2x - 3)(3x + 2)$ is equal to

**A** $6x^2 - 5x - 1$ **B** $6x^2 - 5x - 6$ **C** $6x^3 - 5x^2 - x$
**D** $5x^3 - 5x^2 - 6x$ **E** $6x^3 - 5x^2 - 6x$

**7** Which of the following expressions is not a perfect square?

**A** $4x^2 - 20x + 25$ **B** $p^2 + 6py + 9y^2$ **C** $9x^2 - 12xy + 16y^2$
**D** $4a^2 + 4ab + b^2$ **E** $m^2 - 10mn + 25n^2$

## Short-answer questions

**8** Which of the following expressions are

**i** perfect squares?
**ii** a difference of squares?

**a** $x^2 + 8x + 16$ **b** $a^2 - 1$ **c** $4b^2 - 12b + 9$
**d** $(w - 5)(w + 5)$ **e** $-z^2 + 25$

**9** Expand and simplify each of the following.

**a** $(2a + 3y)(a - y)$
**b** $(2x - y)^2$
**c** $(2x - 5)^2 - (x + 2)(3x + 7)$

**10** Factorise each of the following by taking out a common factor.

**a** $12a^2b^5 - 3ab^8$ **b** $4(2x + 3) - (2x + 3)^2$
**c** $(3y - 1)^2 + (2y + 3)(3y - 1)$ **d** $(2x + y)^2 - 6x - 3y$

**11** Use your knowledge of perfect squares to factorise each of the following.

**a** $x^2 - 14x + 49$ **b** $x^2 + 6xy + 9y^2$

**12** Factorise the following differences of squares.

**a** $144 - c^2$ **b** $9x^2 - 121$
**c** $(x + 3)^2 - 16$ **d** $25 - (a - 4)^2$
**e** $16(x - 3)^2 - (x - 2)^2$ **f** $4x^2 - 3$ (over $R$)

**13** Factorise each of the following.

**a** $5e - 24 + e^2$ **b** $15p^2 - 14p - 8$ **c** $x^2 + 15x + 56$ **d** $x^2 + 2x - 63$
**e** $a^2 + 8a - 48$ **f** $m^2 + 18m + 45$ **g** $x^2 + 4x - 21$ **h** $x^2 - 10x + 21$

**14** Factorise the following expressions by completing the square.

**a** $x^2 + 8x + 1$ **b** $m^2 - 5m + 3$

**15** Factorise each of the following expressions by first taking out the common factor.

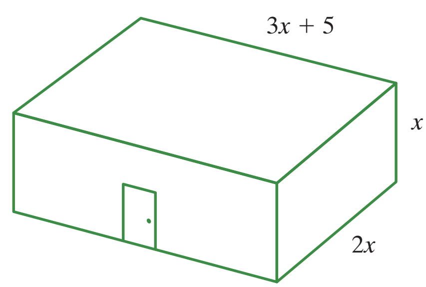
**a** $5x^2 + 10x + 5$ **b** $2a^2 + 24a + 72$ **c** $6x^2 - 24x + 24$
**d** $3m^2 - 42m + 147$ **e** $20x^2 + 120x + 180$ **f** $-3x^2 + 30x - 75$
**g** $-4x^2 - 88x - 484$ **h** $-5x^2 + 90x - 405$ **i** $20x^2 + 60x + 45$
**j** $3a^2 - 12$ **k** $16 - 4x^2$ **l** $4y^2 - 100z^2$
**m** $64d^2 - 16e^2$ **n** $5x^2 - 45$ **o** $12x^2 - 75y^2$

**16** Use any appropriate methods to factorise the following expressions.

**a** $(4x + 1)^2 - (x - 3)^2$ **b** $2m^2 - 4m - 30$
**c** $8(5m + 3)^2 - 14(5m + 3) + 3$ **d** $25x^2 - 30xy + 9y^2$
**e** $y^2 - \frac{2}{5}y + \frac{1}{25}$ **f** $9a^2 - 12ab + 4b^2$
**g** $10w^2 - 35w - 75$ **h** $2(x - 3)^2 - 2(x - 3) - 24$

## Extended-response question

**17** A garden shed made of aluminium panels is designed in the shape of a rectangular prism as shown. The shed does not include a floor. Dimensions are in metres. Find




**a** an expanded expression for the volume of the shed.
**b** an expanded expression for the total surface area of the shed.
**c** the cost of the aluminium if $x = 3$ and the panels cost \$4.70 per square metre.