Chapter 3 Pythagoras' theorem and trigonometry

What you will learn

Australian curriculum

MEASUREMENT AND GEOMETRY

Pythagoras and Trigonometry

Investigate Pythagoras' theorem and its application to solving simple problems involving right-angled triangles Use similarity to investigate the constancy of the sine, cosine and tangent ratios for a given angle in right-angled triangles Apply trigonometry to solve right-angled triangle problems

Online resources

- Chapter pre-test
- Videos of all worked examples
- Interactive widgets
- Interactive walkthroughs
- Downloadable HOTsheets
- Access to HOTmaths Australian Curriculum courses

Satellites

Satellite navigation systems work by determining where you are and calculating how far it is to where you want to go. Distances are worked out using the mathematics of trigonometry. The position of the satellite, your position and your destination are three points which form a triangle. This triangle can be divided into two right-angled triangles and, using two known angles and one side length, the distance between where you are and your destination can be found using sine, cosine and tangent functions. Similar techniques are used to navigate the seas, study the stars and map our planet, Earth.

3A Pythagoras' theorem

Interactive
Widgets
HOTsheets
Walkthroughs

Pythagoras was born on the Greek island of Samos in the 6th century BCE. He received a privileged education and travelled to Egypt and Persia where he developed his ideas in mathematics and philosophy. He settled in Crotone Italy where he founded a school. His many students and followers were called the Pythagoreans and under the guidance of Pythagoras, lived a very structured life with strict rules. They aimed to be pure, self-sufficient and wise, where men and women were treated equally and all property was considered communal. They strove to perfect their physical and mental form and made many advances in their understanding of the world through mathematics.

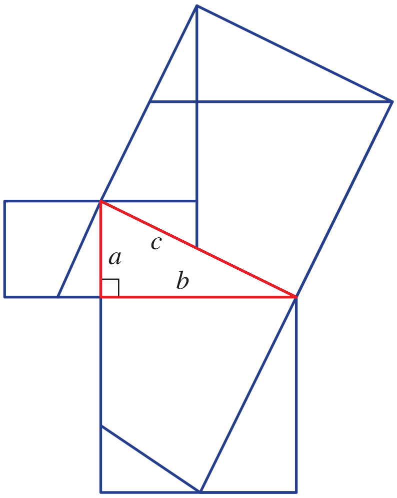

The Pythagorean brotherhood in ancient Greece

The Pythagoreans discovered the famous theorem, which is named after Pythagoras, and the existence of irrational numbers such as $\sqrt{2}$, which cannot be written down as a fraction or terminating decimal. Such numbers cannot be measured exactly with a ruler with fractional parts and were thought to be unnatural. The Pythagoreans called these numbers 'unutterable' numbers and it is believed that any member of the brotherhood who mentioned these numbers in public would be put to death.

Let's start: Matching the areas of squares

Look at this right-angled triangle and the squares drawn on each side. Each square is divided into smaller sections.

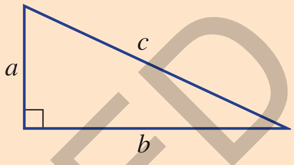
- Can you see how the parts of the two smaller squares would fit into the larger square?
- What is the area of each square if the side lengths of the right-angled triangle are a, b and c as marked?
- What do the answers to the above two questions suggest about the relationship between a, b and c?



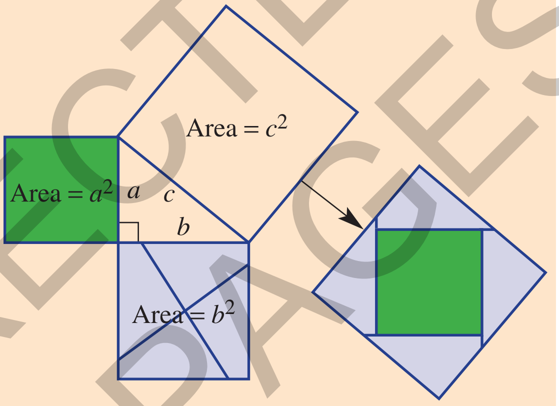
Key ideas

- The longest side of a right-angled triangle is called the **hypotenuse** and is opposite the right angle.
- The **theorem of Pythagoras** says that the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.
  For the triangle shown, it is:

  $$\underbrace{c^2}_{\text{square of the hypotenuse}} = \underbrace{a^2 + b^2}_{\text{squares of the two shorter sides}}$$



  - The theorem can be illustrated in a diagram like the one on the right. The sum of the areas of the two smaller squares ($a^2 + b^2$) is the same as the area of the largest square (c^2).

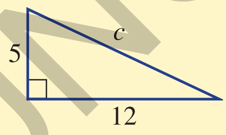


- Lengths can be expressed with **exact values** using **surds**. $\sqrt{2}$, $\sqrt{28}$ and $2\sqrt{3}$ are examples of surds.
  - When expressed as a decimal, a surd is an infinite non-recurring decimal with no pattern.
    For example: $\sqrt{2} = 1.4142135623\ldots$

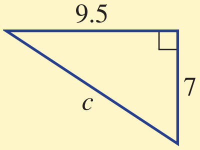
Example 1 Finding the length of the hypotenuse

Find the length of the hypotenuse in these right-angled triangles. Round to two decimal places in part **b**.

a



b



SOLUTION	EXPLANATION
a $c^2 = a^2 + b^2$ $= 5^2 + 12^2$ $= 169$	Write the rule and substitute the lengths of the two shorter sides.
$\therefore c = \sqrt{169}$ $= 13$	If $c^2 = 169$ then $c = \sqrt{169} = 13$.

b $c^2 = a^2 + b^2$

$= 7^2 + 9.5^2$

$= 139.25$

$\therefore\ c = \sqrt{139.25}$

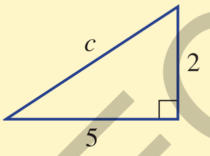
$= 11.80$ (to 2 d.p.)

The order for a and b does not matter since $7^2 + 9.5^2 = 9.5^2 + 7^2$.

Round as required.

Example 2 Finding the length of the hypotenuse using exact values

Find the length of the hypotenuse in this right-angled triangle, leaving your answer as an exact value.



SOLUTION

$c^2 = a^2 + b^2$

$= 5^2 + 2^2$

$= 29$

$\therefore\ c = \sqrt{29}$

EXPLANATION

Apply Pythagoras' theorem to find the value of c.

Express the answer exactly using a surd.

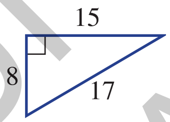
Exercise 3A

1–3	3(½)	—

UNDERSTANDING

1 State the length of the hypotenuse (c units) in these right-angled triangles.

a



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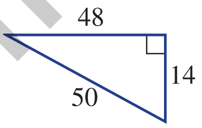
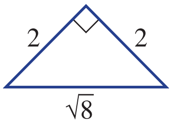
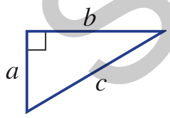


c



2 Write down Pythagoras' theorem using the given pronumerals for these right-angled triangles. For example: $z^2 = x^2 + y^2$.

a



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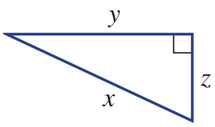

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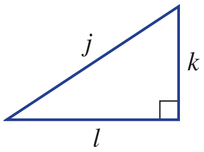


3 Evaluate the following, rounding to two decimal places in parts **g** and **h**.

a 9^2 **b** 3.2^2 **c** $3^2 + 2^2$ **d** $9^2 + 5^2$

e $\sqrt{36}$ **f** $\sqrt{64 + 36}$ **g** $\sqrt{24}$ **h** $\sqrt{3^2 + 2^2}$

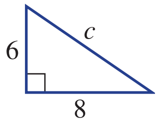
4–7(½) 4–8(½) 4–8(½)

FLUENCY

Example 1a

4 Find the length of the hypotenuse in each of the following right-angled triangles.

a



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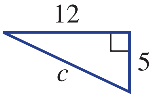

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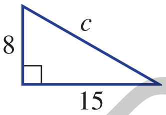
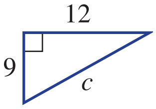
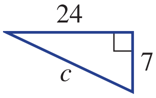


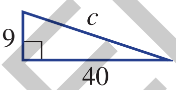
d



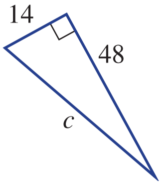
e



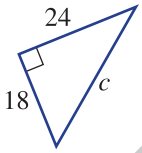
f



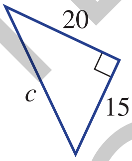
g



h



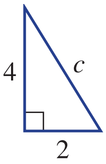
i



Example 1b

5 Find the length of the hypotenuse in each of these right-angled triangles, correct to two decimal places.

a



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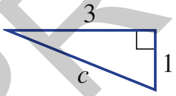
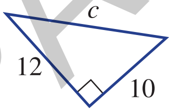
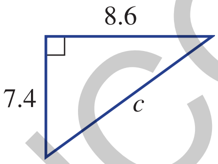


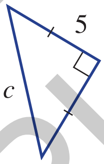
c



d



e



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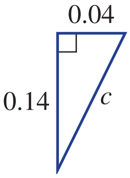
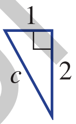


Example 2

6 Find the length of the hypotenuse in these triangles, leaving your answer as an exact value.

a



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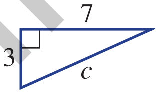

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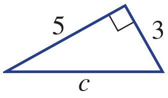

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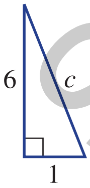

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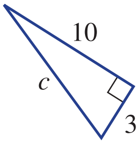

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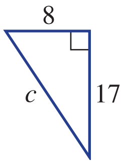
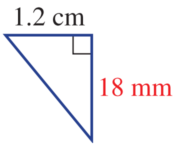


3A

FLUENCY

7 Find the length of the hypotenuse in each of these right-angled triangles, rounding to two decimal places where necessary. Convert to the units indicated in red.

a



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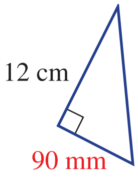

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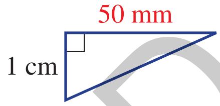

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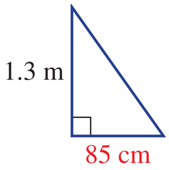

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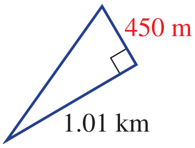

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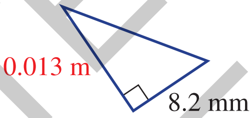
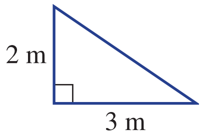


8 For each of these triangles, first calculate the length of the hypotenuse then find the perimeter, correct to two decimal places.

a



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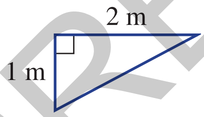

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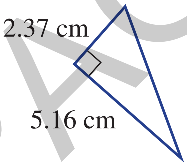

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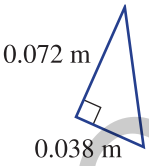

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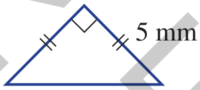

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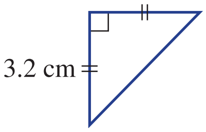
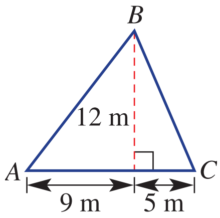


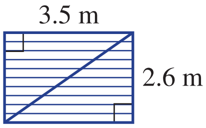
9–11	9–11, 13	11–14

PROBLEM-SOLVING

9 Find the perimeter of this triangle. (*Hint*: You will need to find *AB* and *BC* first.)



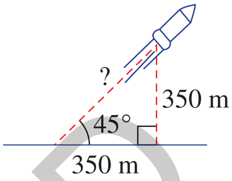
10 Find the length of the diagonal steel brace required to support a wall of length 3.5 m and height 2.6 m. Give your answer correct to one decimal place.



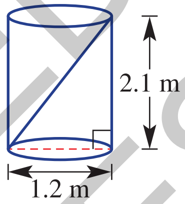
11 A helicopter hovers at a height of 150 m above the ground and is a horizontal distance of 200 m from a beacon on the ground. Find the direct distance of the helicopter from the beacon.

PROBLEM-SOLVING

12 A miniature rocket blasts off at an angle of 45° and travels in a straight line. After a few seconds, reaches a height of 350 m above the ground. At this point it has also covered a horizontal distance of 350 m. How far has the rocket travelled to the nearest metre?



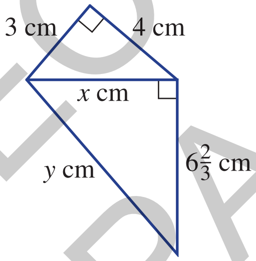
13 Find the length of the longest rod that will fit inside a cylinder of height 2.1 m and with circular end surface of 1.2 m diameter. Give your answer correct to one decimal place.



14 For the shape on the right, find the value of:

a x

b y (as a fraction)



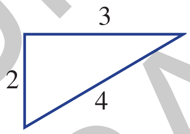
15	15, 16	16, 17

REASONING

15 One way to check whether a four-sided figure is a rectangle is to ensure that both its diagonals are the same length. What should the length of the diagonals be if a rectangle has side lengths 3 m and 5 m? Answer to two decimal places.

16 We know that if the triangle has a right angle, then $c^2 = a^2 + b^2$. The converse of this is that if $c^2 = a^2 + b^2$ then the triangle must have a right angle. Test if $c^2 = a^2 + b^2$ to see if these triangles must have a right angle. They may not be drawn to scale.

a



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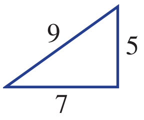

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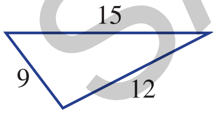
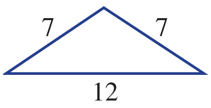
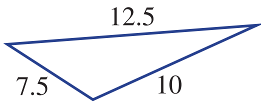


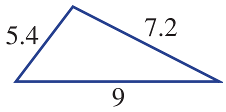
d



e

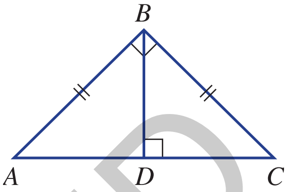


f



17 Triangle ABC is a right-angled isosceles triangle, and BD is perpendicular to AC. If $DC = 4$ cm and $BD = 4$ cm:

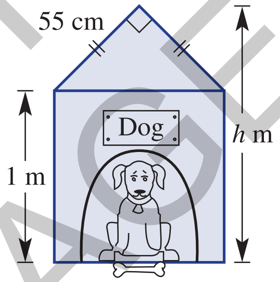
- **a** find the length of BC correct to two decimal places
- **b** state the length of AB correct to two decimal places
- **c** use Pythagoras' theorem and ΔABC to check that the length of AC is twice the length of DC.



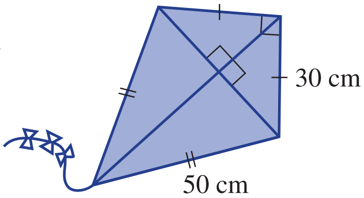
Kennels and kites — — 18, 19

18 A dog kennel has the dimensions shown in the diagram on the right. Give your answers to each of the following correct to two decimal places.

- **a** What is the width, in cm, of the kennel?
- **b** What is the total height, h m, of the kennel?
- **c** If the sloping height of the roof was to be reduced from 55 cm to 50 cm, what difference would this make to the total height of the kennel? (Assume that the width is the same as in part **a**.)
- **d** What is the length of the sloping height of the roof of a new kennel if it is to have a total height of 1.2 m? (The height of the kennel without the roof is still 1 m and its width is unchanged.)



19 The frame of a kite is constructed with six pieces of timber dowel. The four pieces around the outer edge are two 30 cm pieces and two 50 cm pieces. The top end of the kite is to form a right angle. Find the length of each of the diagonal pieces required to complete the construction. Answer to two decimal places.



3B Finding the shorter sides

Interactive

Widgets

HOTsheets

Walkthroughs

Throughout history, mathematicians have utilised known theorems to explore new ideas, discover new theorems and solve a wider range of problems. Similarly, Pythagoras knew that his right-angled triangle theorem could be manipulated so that the length of one of the shorter sides of a triangle can be found if the length of the other two sides are known.

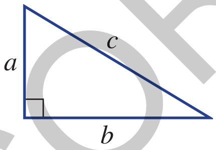
We know that the sum $7 = 3 + 4$ can be written as a difference $3 = 7 - 4$. Likewise, if $c^2 = a^2 + b^2$ then $a^2 = c^2 - b^2$ or $b^2 = c^2 - a^2$.

Applying this to a right-angled triangle means that we can now find the length of one of the shorter sides if the other two sides are known.

If we know the length of the crane jib and the horizontal distance it extends, Pythagoras' theorem enables us to calculate its vertical height.

Let's start: True or false

Below are some mathematical statements relating to a right-angled triangle with hypotenuse c and the two shorter sides a and b.



Some of these mathematical statements are true and some are false. Can you sort them into true and false groups?

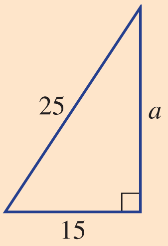
$a^2 + b^2 = c^2$	$a = \sqrt{c^2 - b^2}$	$c^2 - a^2 = b^2$	$a^2 - c^2 = b^2$
$c = \sqrt{a^2 + b^2}$	$b = \sqrt{a^2 - c^2}$	$c = \sqrt{a^2 - b^2}$	$c^2 - b^2 = a^2$

Key ideas

- When finding the length of a side:
  - substitute known values into Pythagoras' rule
  - solve this equation to find the unknown value.

  For example:
  - If $a^2 + 16 = 30$ then subtract 16 from both sides.
  - If $a^2 = 14$ then take the square root of both sides.
  - $a = \sqrt{14}$ is an **exact** answer (a surd).
  - $a = 3.74$ is a rounded decimal answer.

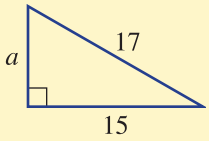
$$\begin{aligned} c^2 &= a^2 + b^2 \\ 25^2 &= a^2 + 15^2 \\ 625 &= a^2 + 225 \\ 400 &= a^2 \\ a &= 20 \end{aligned}$$



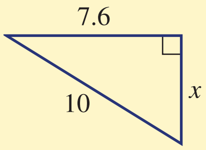
Example 3 Finding the length of a shorter side

In each of the following, find the value of the pronumeral. Round your answer in part **b** to two decimal places and give an exact answer to part **c**.

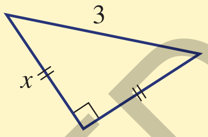
a



b



c



SOLUTION	EXPLANATION
a $a^2 + 15^2 = 17^2$	Write the rule and substitute the known sides.
$a^2 + 225 = 289$	Square 15 and 17.
$a^2 = 64$	Subtract 225 from both sides.
$\therefore a = \sqrt{64}$	Take the square root of both sides.
$a = 8$	
b $x^2 + 7.6^2 = 10^2$	Write the rule.
$x^2 + 57.76 = 100$	
$x^2 = 42.24$	Subtract 57.76 from both sides.
$\therefore x = \sqrt{42.24}$	Take the square root of both sides.
$x = 6.50$ (to 2 d.p.)	Round to two decimal places.
c $x^2 + x^2 = 3^2$	Two sides are of length x.
$2x^2 = 9$	Add like terms and then divide both sides by 2.
$x^2 = \frac{9}{2}$	Take the square root of both sides. To express as an exact answer, do not round.
$\therefore x = \sqrt{\frac{9}{2}} \left(= \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2} \right)$	Different forms are possible.

Exercise 3B

1(½), 2	2	—

UNDERSTANDING

1 Find the value of a or b in these equations. (Both a and b are positive numbers.)

- **a** $a = \sqrt{196}$
- **b** $a = \sqrt{121}$
- **c** $a^2 = 144$
- **d** $a^2 = 400$
- **e** $b^2 + 9 = 25$
- **f** $b^2 + 49 = 625$
- **g** $36 + b^2 = 100$
- **h** $15^2 + b^2 = 289$

2 If $a^2 + 64 = 100$, decide if the following are true or false.

- **a** $a^2 = 100 - 64$
- **b** $64 = 100 + a^2$
- **c** $100 = \sqrt{a^2 + 64}$
- **d** $a = \sqrt{100 - 64}$
- **e** $a = 6$
- **f** $a = 10$

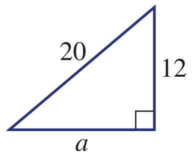
3–5(½)	3–5(½), 6	3–5(½), 6

FLUENCY

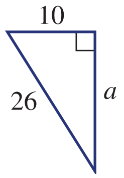
Example 3a

3 In each of the following find the value of the pronumeral.

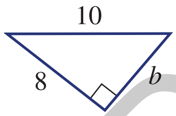
a



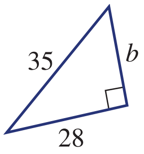
b



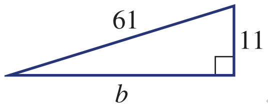
c



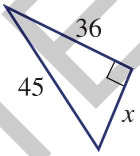
d



e



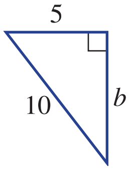
f



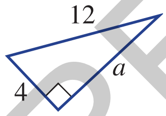
Example 3b

4 In each of the following, find the value of the pronumeral. Express your answers correct to two decimal places.

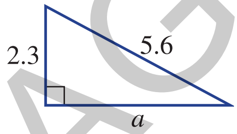
a



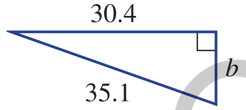
b



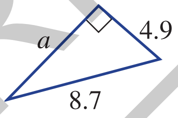
c



d



e



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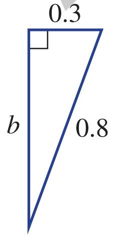
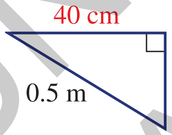
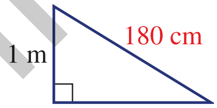


5 Find the length of the unknown side of each of these triangles, correct to two decimal places where necessary. Convert to the units shown in red.

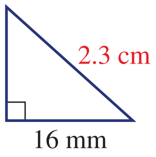
a



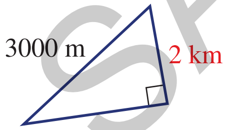
b



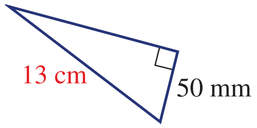
c



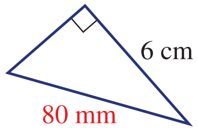
d



e



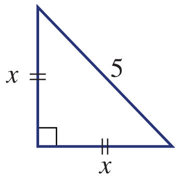
f

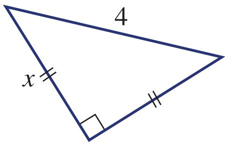


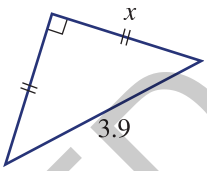
FLUENCY

Example 3c

6 In each of the following, find the value of x as an exact answer.

a


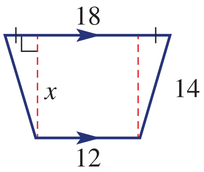
b


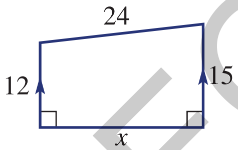
c


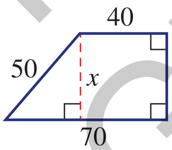
7–9	7, 8, 10	9–11

PROBLEM-SOLVING

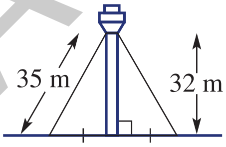
7 For each of the following diagrams, find the value of x. Give an exact answer each time.

a


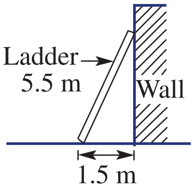
b


c


8 A 32 m communication tower is supported by 35 m cables stretching from the top of the tower to a position at ground level. Find the distance from the base of the tower to the point where the cable reaches the ground, correct to one decimal place.

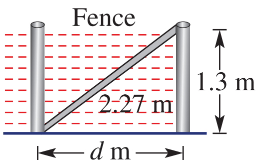


9 The base of a ladder leaning against a vertical wall is 1.5 m from the base of the wall. If the ladder is 5.5 m long, find how high the top of the ladder is above the ground, correct to one decimal place.



10 If a television has a screen size of 63 cm it means that the diagonal length of the screen is 63 cm. If the vertical height of a 63 cm screen is 39 cm, find how wide the screen is to the nearest centimetre.

11 A 1.3 m vertical fence post is supported by a 2.27 m bar, as shown in the diagram on the right. Find the distance (d metres) from the base of the post to where the support enters the ground. Give your answer correct to two decimal places.

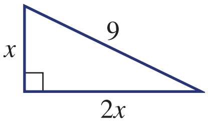


12	12	12, 13

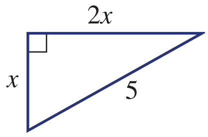
REASONING

12 For these questions note that $(2x)^2 = 4x^2$ and $(3x)^2 = 9x^2$.
In each of the following find the value of x as an exact answer.

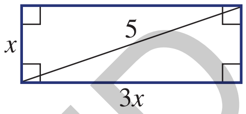
a



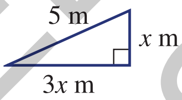
b



c



13 A right-angled triangle has a hypotenuse measuring 5 m. Find the lengths of the other sides if their lengths are in the given ratio. Give an exact answer.
Hint: You can draw a triangle like the one shown for part **a**.



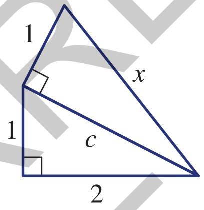
a 1 to 3 **b** 2 to 3 **c** 5 to 7

The power of exact values

—	—	14, 15

ENRICHMENT

14 Consider this diagram and the unknown length x.



a Explain what needs to be found first before x can be calculated.
b Now try calculating the value x as an exact value.
c Was it necessary to calculate the value of c or was c^2 enough?
d What problems might be encountered if the value of c was calculated and rounded before the value of x is found?

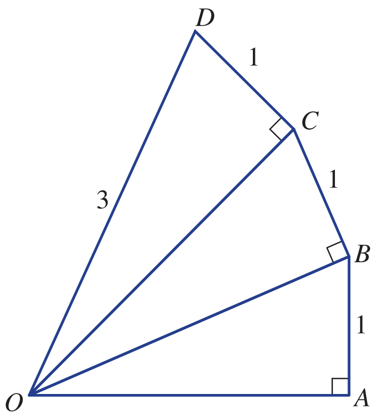
15 In the diagram below, $OD = 3$ and $AB = BC = CD = 1$.

a Using exact values find the length of:
  i OC **ii** OB **iii** OA

b Round your answer in part **a iii** to one decimal place and use that length to recalculate the lengths of OB, OC and OD (correct to two decimal places) starting with ΔOAB.

c Explain the difference between the given length $OD = 3$ and your answer for OD in part **b**.

d Investigate how changing the side length AB affects your answers to parts **a** to **c** above.



3C Applying Pythagoras' theorem

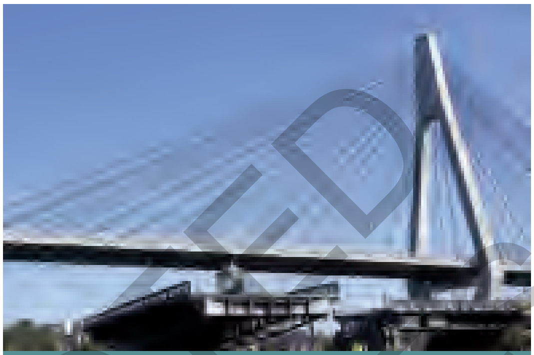
Interactive

Widgets

HOTsheets

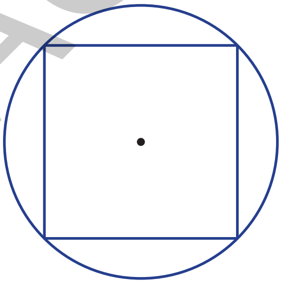
Walkthroughs

Initially it may not be obvious that Pythagoras' theorem can be used to help solve a particular problem. With further investigation, however, it may be possible to identify and draw in a right-angled triangle which can help solve the problem. As long as two sides of the right-angled triangle are known, the length of the third side can be found.


The length of each cable on the Anzac Bridge, Sydney can be calculated using Pythagoras' theorem.

Let's start: The biggest square

Imagine trying to cut the largest square from a circle of a certain size and calculating the side length of the square. Drawing a simple diagram as shown does not initially reveal a right-angled triangle.



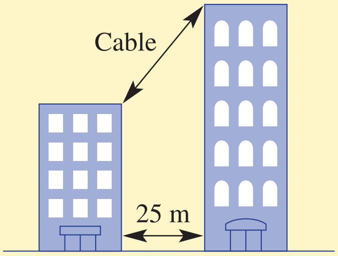
- If the circle has a diameter of 2 cm, can you find a good position to draw the diameter that also helps to form a right-angled triangle?
- Can you determine the side length of the largest square?
- What percentage of the area of a circle does the largest square occupy?

Key ideas

- When applying Pythagoras' theorem, follow these steps.
  - Identify and draw right-angled triangles which may help to solve the problem.
  - Label the sides with their lengths or with a letter (pronumeral) if the length is unknown.
  - Use Pythagoras' theorem to solve for the unknown.
  - Solve the problem by making any further calculations and answering in words.

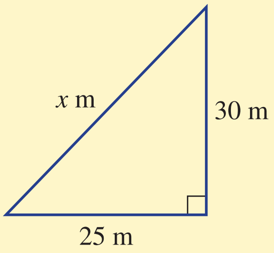
Example 4 Applying Pythagoras' theorem

Two skyscrapers are located 25 m apart and a cable links the tops of the two buildings. Find the length of the cable if the buildings are 50 m and 80 m in height. Give your answer correct to two decimal places.



SOLUTION

Let x m be the length of the cable.

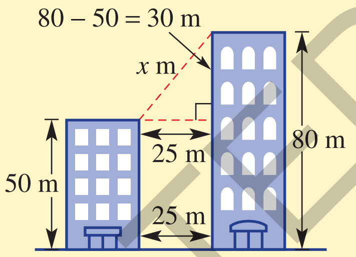


$$\begin{aligned} c^2 &= a^2 + b^2 \\ x^2 &= 25^2 + 30^2 \\ x^2 &= 625 + 900 \\ &= 1525 \\ \therefore x &= \sqrt{1525} \\ &= 39.05 \end{aligned}$$

$\therefore$ The cable is 39.05 m long.

EXPLANATION

Draw a right-angled triangle and label the measurements and pronumerals.



Set up an equation using Pythagoras' theorem and solve for x.

Answer the question in words.

Exercise 3C

1 1 —

UNDERSTANDING

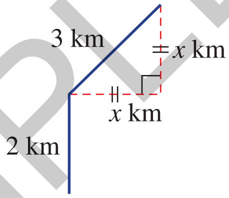
1 Match each problem (**a**, **b** or **c**) with both a diagram (**A**, **B** or **C**) and its solution (**I**, **II**, **III**).

a Two trees stand 20 m apart and they are 32 m and 47 m tall. What is the distance between the tops of the two trees?

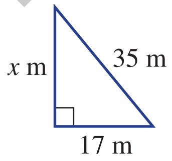
b A man walks due north for 2 km then north-east for 3 km. How far north is he from his starting point?

c A kite is flying with a kite string of length 35 m. Its horizontal distance from its anchor point is 17 m. How high is the kite flying?

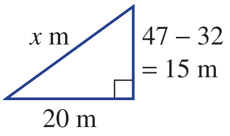
A



B



C



I The kite is flying at a height of 30.59 m.

II The distance between the top of the two trees is 25 m.

III The man has walked a total of 2 + 2.12 = 4.12 km north from his starting point.

3C

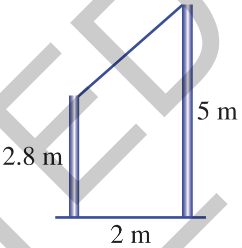
2–5	2, 4, 6	2, 4, 6(½)

FLUENCY

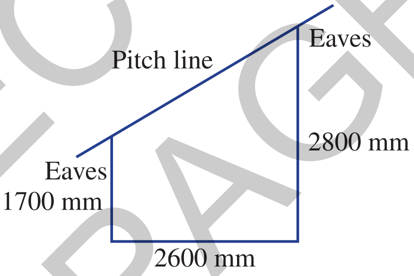
Example 4

2 Two skyscrapers are located 25 m apart and a cable of length 62.3 m links the tops of the two buildings. If the taller building is 200 metres tall, what is the height of the shorter building? Give your answer correct to one decimal place.

3 Two poles are located 2 m apart. A wire links the tops of the two poles. Find the length of the wire if the poles are 2.8 m and 5 m in height. Give your answer correct to one decimal place.



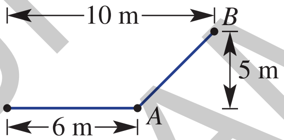
4 A garage is to be built with a skillion roof (a roof with a single slope). The measurements are given in the diagram. Calculate the pitch line length, to the nearest millimetre. Allow 500 mm for each of the eaves.



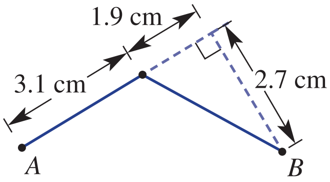
5 Two bushwalkers are standing on different mountain sides. According to their maps, one of them is at a height of 2120 m and the other is at a height of 1650 m. If the horizontal distance between them is 950 m, find the direct distance between the two bushwalkers. Give your answer correct to the nearest metre.

6 Find the direct distance between the points *A* and *B* in each of the following, correct to one decimal place.

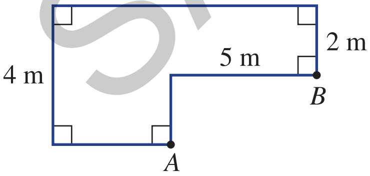
a



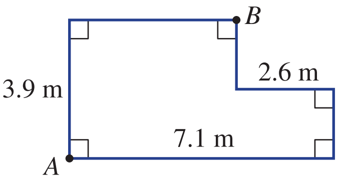
b



c

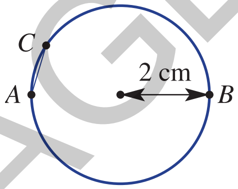


d



7 A 100 m radio mast is supported by six cables in two sets of three cables. They are anchored to the ground at an equal distance from the mast. The top set of three cables is attached at a point 20 m below the top of the mast. Each cable in the lower set of three cables is 60 m long and is attached at a height of 30 m above the ground. If all the cables have to be replaced, find the total length of cable required. Give your answer correct to two decimal places.

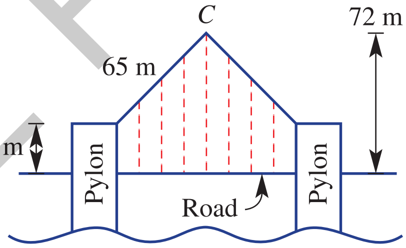
8 In a particular circle of radius 2 cm, AB is a diameter and C is a point on the circumference. Angle ACB is a right angle. The chord AC is 1 cm in length.



a Draw the triangle ABC as described, and mark in all the important information.

b Find the length of BC correct to one decimal place.

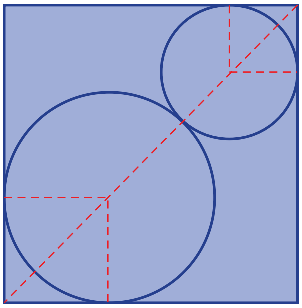
9 A suspension bridge is built with two vertical pylons and two straight beams of equal length that are positioned to extend from the top of the pylons to meet at a point C above the centre of the bridge, as shown in the diagram on the right.



a Calculate the vertical height of the point C above the tops of the pylons.

b Calculate the distance between the pylons, that is, the length of the span of the bridge correct to one decimal place.

10 Two circles of radii 10 cm and 15 cm respectively are placed inside a square. Find the perimeter of the square to the nearest centimetre. *Hint*: first find the diagonal length of the square using the diagram on the right.



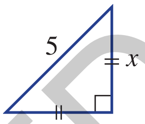
3C

11	11	11, 12

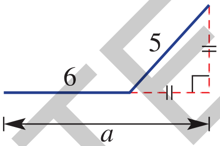
REASONING

11 It is possible to find the length of the shorter sides of a right-angled isosceles triangle if only the hypotenuse length is known.

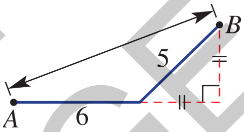
a Find the exact value of x in this right-angled isosceles triangle.



b Now find the exact value of a in this diagram.



c Finally, use your results from above to find the length of AB in this diagram correct to one decimal place.



12 Use the method outlined in Question **11** for this problem.

In an army navigation exercise, a group of soldiers hiked due south from base camp for 2.5 km to a water hole. From there, they turned 45° to the left, to head south-east for 1.6 km to a resting point. When the soldiers were at the resting point, how far (correct to one decimal place):

a east were they from the water hole?

b south were they from the water hole?

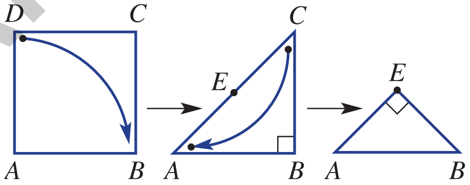
c were they in a straight line from base camp?

Folding paper

—	—	13

ENRICHMENT

13 A square piece of paper, $ABCD$, of side length 20 cm is folded to form a right-angled triangle ABC. The paper is folded a second time to form a right-angled triangle ABE as shown in the diagram below.



a Find the length of AC correct to two decimal places.

b Find the perimeter of each of the following, correct to one decimal place where necessary:

- **i** square $ABCD$
- **ii** triangle ABC
- **iii** triangle ABE

c Use Pythagoras' theorem and your answer for part **a** to confirm that $AE = BE$ in triangle ABE.

d Investigate how changing the initial side length changes the answers to the above.

3D Pythagoras in three dimensions EXTENDING

Interactive
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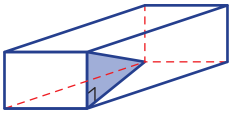
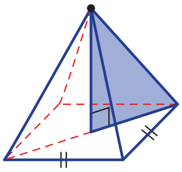
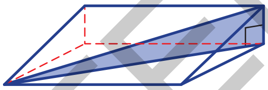
Walkthroughs

If you cut a solid to form a cross-section a two-dimensional shape is revealed. From that cross-section it may be possible to identify a right-angled triangle that can be used to find unknown lengths. These lengths can then tell us information about the three-dimensional solid.

You can visualise right-angled triangles in all sorts of different solids.



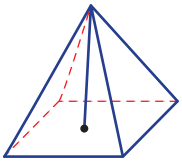




The glass pyramid at the Palais du Louvre, Paris, is made up of a total of 70 triangular and 603 rhombus-shaped glass segments together forming many right-angled triangles.

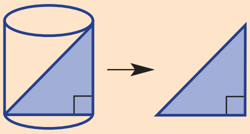
Let's start: How many triangles in a pyramid?

Here is a drawing of a square-based pyramid. By drawing lines from any vertex to the centre of the base and another point, how many different right-angled triangles can you visualise and draw? The triangles could be inside or on the outside surface of the pyramid.



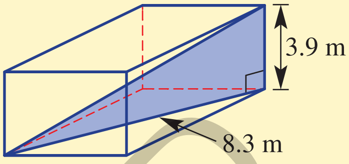
Key ideas

- Right-angled triangles can be identified in many three-dimensional solids.
- It is important to try to draw any identified right-angled triangle using a separate diagram.

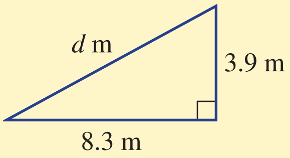


Example 5 Using Pythagoras in 3D

The length of the diagonal on the base of a rectangular prism is 8.3 m and the rectangular prism's height is 3.9 m. Find the distance from one corner of the rectangular prism to the opposite corner. Give your answer correct to two decimal places.



SOLUTION	EXPLANATION
Let d m be the distance required.	Draw a right-angled triangle and label all the measurements and pronumerals.



$$d^2 = 3.9^2 + 8.3^2$$
$$= 84.1$$
$$\therefore d = 9.17$$

Use Pythagoras' theorem.
Round to two decimal places.

The distance from one corner of the rectangular prism to the opposite corner is approximately 9.17 m.

Write your answer in words.

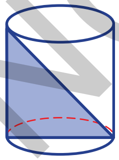
Exercise 3D

1	1	—

UNDERSTANDING

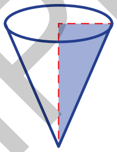
1 Decide if the following shaded regions would form right-angled triangles.

a



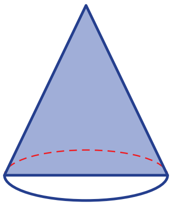
Right cylinder

b



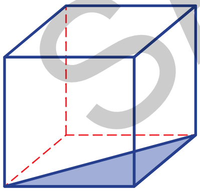
Right cone

c



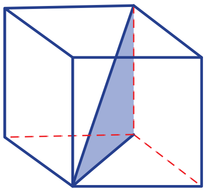
Cone

d



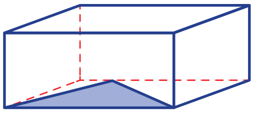
Cube

e



Cube

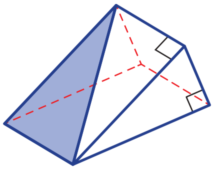
f



Rectangular prism

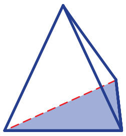
UNDERSTANDING

g



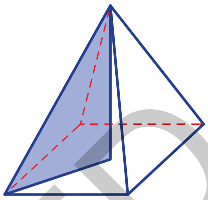
Triangular prism

h



Tetrahedron (regular triangular-based pyramid)

i



Right square-based pyramid (apex above centre of base)

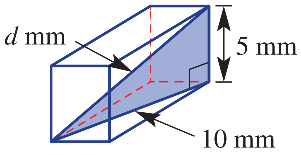
2–5 | 2–6 | 2–4(½), 5, 6

FLUENCY

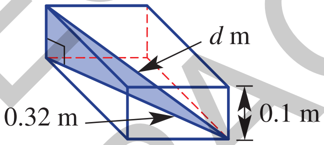
Example 5

2 Find the distance, *d* units, from one corner to the opposite corner in each of the following rectangular prisms. Give your answers correct to two decimal places.

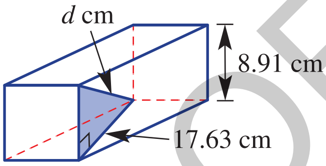
a



b

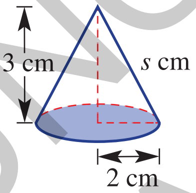


c

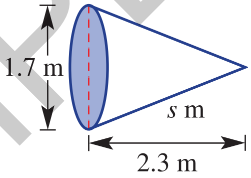


3 Find the slant height, *s* units, of each of the following cones. Give your answers correct to one decimal place.

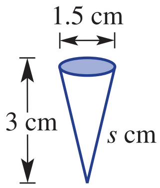
a



b

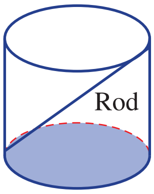


c



4 Find the length to the nearest millimetre of the longest rod that will fit inside a cylinder of the following dimensions.

- **a** Diameter 10 cm and height 15 cm
- **b** Radius 2.8 mm and height 4.2 mm
- **c** Diameter 0.034 m and height 0.015 m

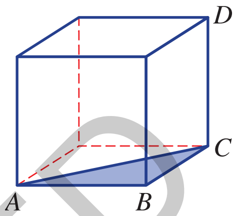


3D

FLUENCY

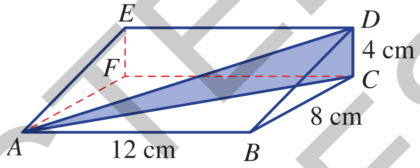
5 The cube in the diagram on the right has 1 cm sides.

- **a** Find the length of AC as an exact value.
- **b** Hence, find the length of AD correct to one decimal place.



6 Consider the shape shown.

- **a** Find the length of AC as an exact value.
- **b** Hence, find the length of AD correct to one decimal place.



7	7, 8	8, 9

PROBLEM-SOLVING

7 A miner makes claim to a circular piece of land with a radius of 40 m from a given point, and is entitled to dig to a depth of 25 m. If the miner can dig tunnels at any angle, find the length of the longest straight tunnel that he can dig, to the nearest metre.

8 A bowl is in the shape of a hemisphere (half sphere) with radius 10 cm. The surface of the water in the container has a radius of 7 cm. How deep is the water? Give your answer to two decimal places.

9 A cube of side length l sits inside a sphere of radius r so that the vertices of the cube sit on the sphere. Find the ratio $r : l$.

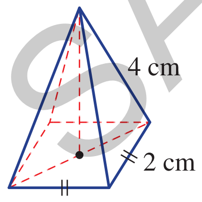
10	10	10, 11

REASONING

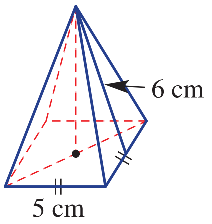
10 There are different ways to approach finding the height of a pyramid depending on what information is given. For each of the following square-based pyramids, find:

- **i** the exact length (using a surd) of the diagonal on the base
- **ii** the height of the pyramid correct to two decimal places.

a



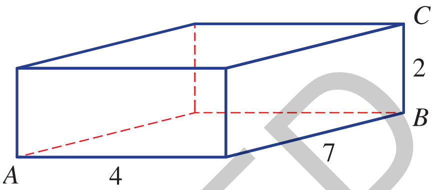
b



REASONING

11 For this rectangular prism answer these questions.

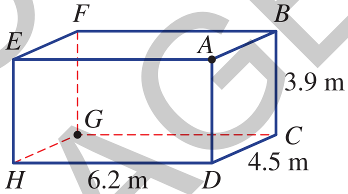
- **a** Find the exact length AB.
- **b** Find AB correct to two decimal places.
- **c** Find the length AC using your result from part **a** and then round to two decimal places.
- **d** Find the length AC using your result from part **b** and then round to two decimal places.
- **e** How can you explain the difference between your results from parts **c** and **d** above?



ENRICHMENT

Spider crawl — — 12

12 A spider crawls from one corner, A, of the ceiling of a room to the opposite corner, G, on the floor. The room is a rectangular prism with dimensions as given in the diagram on the right.



- **a** Assuming the spider crawls in a direct line between points, find how far (correct to two decimal places) the spider crawls if it crawls from A to G via:
  - **i** B **ii** C **iii** D **iv** F
- **b** Investigate other paths to determine the shortest distance that the spider could crawl in order to travel from point A to point G. (*Hint*: consider drawing a net for the solid.)

3E Trigonometric ratios

Interactive
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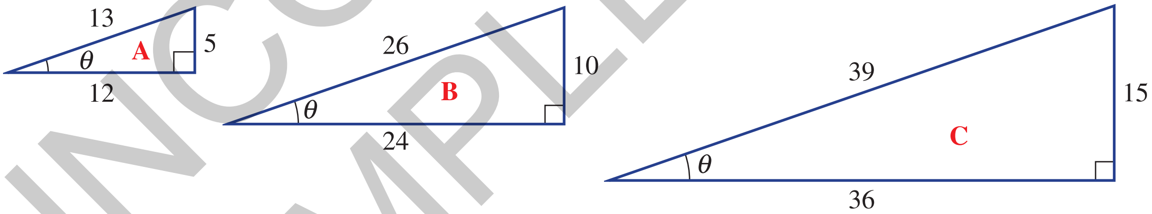
Walkthroughs


A basic form of trigonometry was used in the building of pyramids in ancient Egypt.

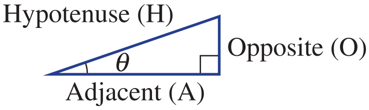
The branch of mathematics called trigonometry deals with the relationship between the side lengths and angles in triangles. Trigonometry dates back to the ancient Egyptian and Babylonian civilisations where a basic form of trigonometry was used in the building of pyramids and in the study of astronomy. The first table of values including chord and arc lengths on a circle for a given angle was created by Hipparchus in the 2nd century BCE in Greece. These tables of values helped to calculate the position of the planets. About three centuries later, Claudius Ptolemy advanced the study of trigonometry writing 13 books called the *Almagest*. Ptolemy also developed tables of values linking the sides and angles of a triangle and produced many theorems which use the sine, cosine and tangent functions.

Let's start: Constancy of sine, cosine and tangent

In geometry we would say that similar triangles have the same shape but are of different size. Here are three similar right-angled triangles. The angle θ (theta) is the same for all three triangles.



We will now calculate three special ratios: sine, cosine and tangent for the angle θ in the above triangles. We use the sides labelled Hypotenuse (H), Opposite (O) and Adjacent (A) as shown at right.

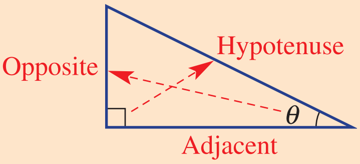


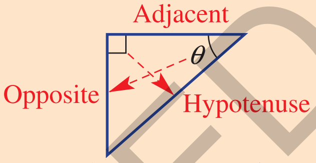
- Complete this table simplifying all fractions.
- What do you notice about the value of:
  - **a** $\sin\theta \left(\text{i.e. } \frac{O}{H}\right)$ for all three triangles?
  - **b** $\cos\theta \left(\text{i.e. } \frac{A}{H}\right)$ for all three triangles?
  - **c** $\tan\theta \left(\text{i.e. } \frac{O}{A}\right)$ for all three triangles?
- Why are the three ratios ($\sin\theta$, $\cos\theta$ and $\tan\theta$) the same for all three triangles? Discuss.

Triangle	$\frac{O}{H}$ ($\sin\theta$)	$\frac{A}{H}$ ($\cos\theta$)	$\frac{O}{A}$ ($\tan\theta$)
A	$\frac{5}{13}$		
B		$\frac{24}{26} = \frac{12}{13}$	
C			$\frac{15}{36} = \frac{5}{12}$

Key ideas

- For a right-angled triangle with another angle named θ:
  - The **hypotenuse** is the longest side, opposite the 90° angle
  - The **opposite** side is opposite θ
  - The **adjacent** side is next to θ but not the hypotenuse.

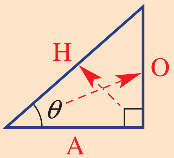




- For a right-angled triangle with a given angle θ, the three ratios **sine (sin)**, **cosine (cos)** and **tangent (tan)** are given by:
  - sine of angle θ (or $\sin\theta$) $= \dfrac{\text{length of the opposite side}}{\text{length of the hypotenuse}}$
  - cosine of angle θ (or $\cos\theta$) $= \dfrac{\text{length of the adjacent side}}{\text{length of the hypotenuse}}$
  - tangent of angle θ (or $\tan\theta$) $= \dfrac{\text{length of the opposite side}}{\text{length of the adjacent side}}$
- For any right-angled triangle with the same angles, these ratios are always the same.
- The term **SOHCAHTOA** is useful when trying to remember the three ratios.

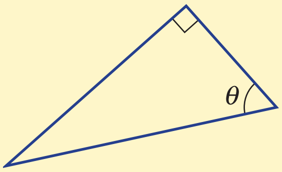
SOH CAH TOA

$\sin\theta = \dfrac{\text{Opposite}}{\text{Hypotenuse}}$ $\cos\theta = \dfrac{\text{Adjacent}}{\text{Hypotenuse}}$ $\tan\theta = \dfrac{\text{Opposite}}{\text{Adjacent}}$

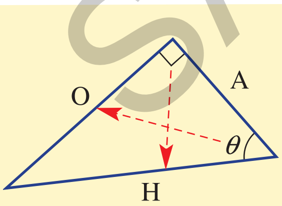


Example 6 Labelling the sides of triangles

Copy this triangle and label the sides as opposite to θ (O), adjacent to θ (A) or hypotenuse (H).



SOLUTION



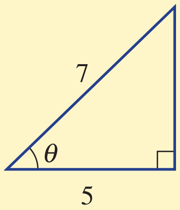
EXPLANATION

Draw the triangle and label the side opposite the right angle as hypotenuse (H), the side opposite the angle θ as opposite (O) and the remaining side next to the angle θ as adjacent (A).

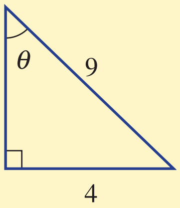
Example 7 Writing trigonometric ratios

Write a trigonometric ratio (in fraction form) for each of the following triangles.

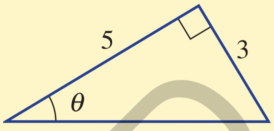
a



b

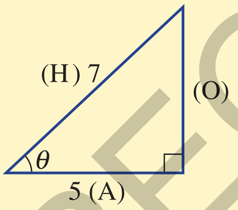


c



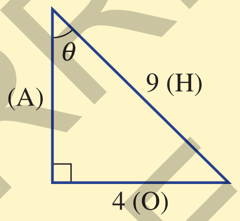
SOLUTION **EXPLANATION**

a $\cos\theta = \frac{A}{H} = \frac{5}{7}$



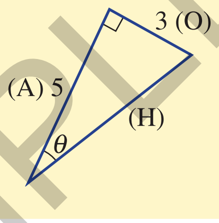
Side length 7 is opposite the right angle so it is the hypotenuse (H). Side length 5 is adjacent to angle θ so it is the adjacent (A).

b $\sin\theta = \frac{O}{H} = \frac{4}{9}$



Side length 9 is opposite the right angle so it is the hypotenuse (H). Side length 4 is opposite angle θ so it is the opposite (O).

c $\tan\theta = \frac{O}{A} = \frac{3}{5}$



Side length 5 is the adjacent side to angle θ so it is the adjacent (A). Side length 3 is opposite angle θ so it is the opposite (O).

Exercise 3E

1, 2(½), 3	3	—

UNDERSTANDING

1 Write the missing word in these sentences.

a H stands for the word ___________.

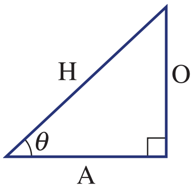
b O stands for the word ___________.

c A stands for the word ___________.

d $\sin\theta =$ ___________ ÷ Hypotenuse.

e $\cos\theta =$ Adjacent ÷ ___________.

f $\tan\theta =$ Opposite ÷ ___________.

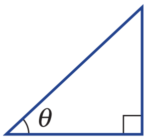


UNDERSTANDING

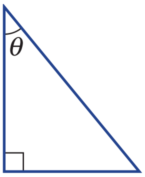
Example 6

2 Copy each of these triangles and label the sides as opposite to θ (O), adjacent to θ (A) or hypotenuse (H).

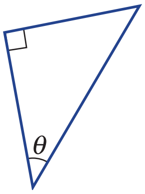
a



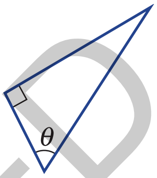
b



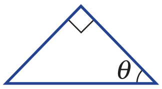
c



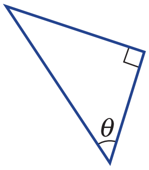
d



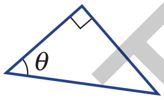
e



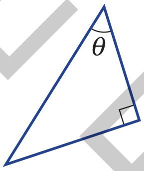
f



g



h



3 For the triangle shown, state the length of the side which corresponds to:

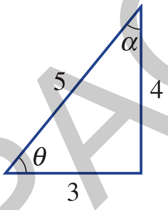
a the hypotenuse

b the side opposite angle θ

c the side opposite angle α

d the side adjacent to angle θ

e the side adjacent to angle α.



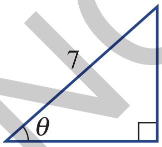
4–6	4(½), 5–7	4(½), 5, 6(½), 7

FLUENCY

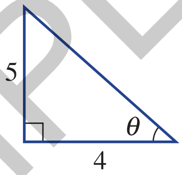
Example 7

4 Write a trigonometric ratio (in fraction form) for each of the following triangles and simplify where possible.

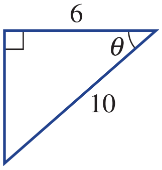
a



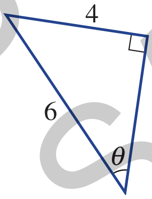
b



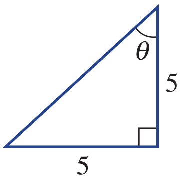
c



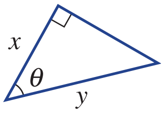
d



e



f



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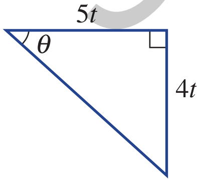

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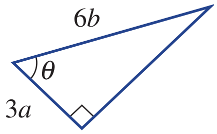
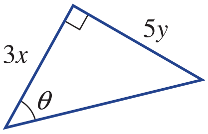
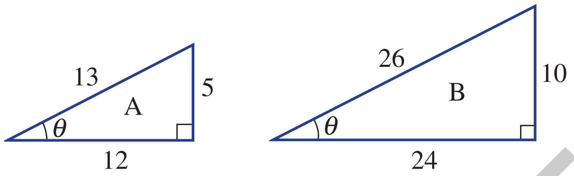


i



5 Here are two similar triangles A and B.



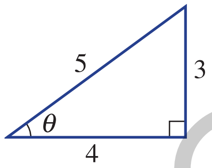
a **i** Write the ratio $\sin\theta$ (as a fraction) for triangle A.
  ii Write the ratio $\sin\theta$ (as a fraction) for triangle B.
  iii What do you notice about your two answers from parts **a i** and **a ii** above?

b **i** Write the ratio $\cos\theta$ (as a fraction) for triangle A.
  ii Write the ratio $\cos\theta$ (as a fraction) for triangle B.
  iii What do you notice about your two answers from parts **b i** and **b ii** above?

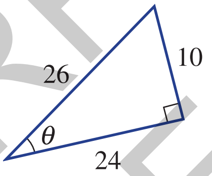
c **i** Write the ratio $\tan\theta$ (as a fraction) for triangle A.
  ii Write the ratio $\tan\theta$ (as a fraction) for triangle B.
  iii What do you notice about your two answers from parts **c i** and **c ii** above?

6 For each of these triangles, write a ratio (in simplified fraction form) for $\sin\theta$, $\cos\theta$ and $\tan\theta$.

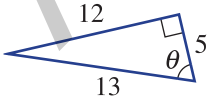
a



b

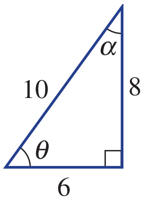


c

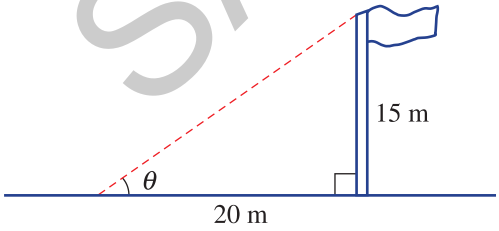


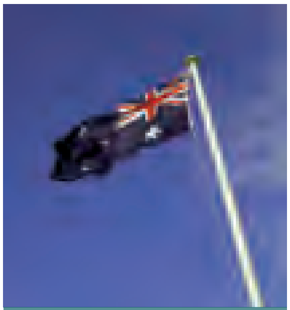
7 For the triangle shown on the right, write a ratio (in fraction form) for:

a $\sin\theta$ **b** $\sin\alpha$ **c** $\cos\theta$
d $\tan\alpha$ **e** $\cos\alpha$ **f** $\tan\theta$



8 A vertical flag pole casts a shadow 20 m long. If the pole is 15 m high, find the ratio for $\tan\theta$.





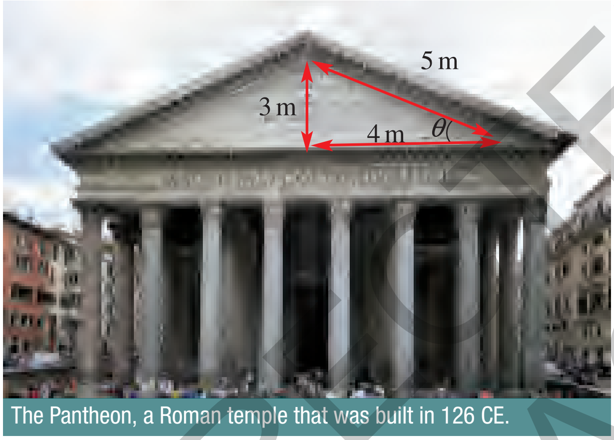
We can use trigonometry to calculate the angle of the shadow that the pole casts.

9 The facade of a Roman temple has the given measurements below. Write down the ratio for:

a sin θ

b cos θ

c tan θ



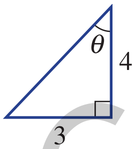
The Pantheon, a Roman temple that was built in 126 CE.

10 For each of the following:

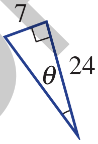
i Use Pythagoras' theorem to find the unknown side.

ii Find the ratios for sin θ, cos θ and tan θ.

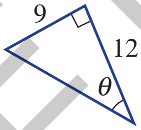
a



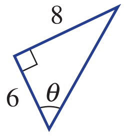
b



c



d



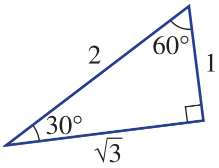
11 **a** Draw a right-angled triangle and mark one of the angles as θ. Mark in the length of the opposite side as 15 units and the length of the hypotenuse as 17 units.

b Using Pythagoras' theorem, find the length of the adjacent side.

c Determine the ratios for sin θ, cos θ and tan θ.

12	12, 13	12–14

12 This triangle has angles 90°, 60° and 30° and side lengths 1, 2 and $\sqrt{3}$.



a Write a ratio for:

i sin 30° **ii** cos 30° **iii** tan 30°

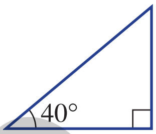
iv sin 60° **v** cos 60° **vi** tan 60°

b What do you notice about the following pairs of ratios?

i cos 30° and sin 60°

ii sin 30° and cos 60°

REASONING



13 **a** Measure all the side lengths of this triangle to the nearest millimetre.

b Use your measurements from part **a** to find an approximate ratio for:

i cos 40° **ii** sin 40° **iii** tan 40°

iv sin 50° **v** tan 50° **vi** cos 50°

c Do you notice anything about the trigonometric ratios for 40° and 50°?

14 Decide if it is possible to draw a right-angled triangle with the given properties. Explain.

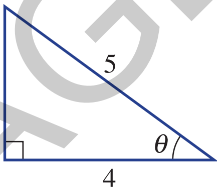
a $\tan \theta = 1$ **b** $\sin \theta = 1$

c $\cos \theta = 0$ **d** $\sin \theta > 1$ or $\cos \theta > 1$

Pythagorean extensions

—	—	15

ENRICHMENT



15 **a** Given that θ is acute and $\cos \theta = \frac{4}{5}$, find $\sin \theta$ and $\tan \theta$.

Hint: use Pythagoras' theorem.

b For each of the following, draw a right-angled triangle then use it to find the other two trigonometric ratios.

i $\sin \theta = \frac{1}{2}$ **ii** $\cos \theta = \frac{1}{2}$ **iii** $\tan \theta = 1$

c Use your results from part **a** to calculate $(\cos \theta)^2 + (\sin \theta)^2$. What do you notice?

d Evaluate $(\cos \theta)^2 + (\sin \theta)^2$ for other combinations of $\cos \theta$ and $\sin \theta$. Research and describe what you have found.

3F Finding side lengths

Interactive
Widgets
HOTsheets
Walkthroughs

For similar triangles we know that the ratio of corresponding sides is always the same. This implies that the three trigonometric ratios for similar right-angled triangles are also constant if the internal angles are equal. Since ancient times, mathematicians have attempted to tabulate these ratios for varying angles. Here are the ratios for some angles in a right-angled triangle, correct to three decimal places.

Angle (θ)	sin θ	cos θ	tan θ
0°	0	1	0
15°	0.259	0.966	0.268
30°	0.5	0.866	0.577
45°	0.707	0.707	1
60°	0.866	0.5	1.732
75°	0.966	0.259	3.732
90°	1	0	undefined


Trigonometric tables in a 400-year old European book.

In modern times these values can be evaluated using calculators to a high degree of accuracy and can be used to help solve problems involving triangles with unknown side lengths.

Let's start: Calculator start-up

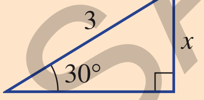
All scientific or CAS calculators can produce accurate values of sin θ, cos θ and tan θ.

- Ensure that your calculator is in degree mode.
- Check the values in the above table to ensure that you are using the calculator correctly.
- Use trial and error to find (to the nearest degree) an angle θ which satisfies these conditions:

a sin $\theta = 0.454$ **b** cos $\theta = 0.588$ **c** tan $\theta = 9.514$

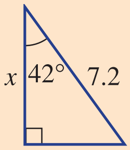
Key ideas

- If θ is in degrees, the ratios for sin θ, cos θ and tan θ can accurately be found using a calculator in **degree mode**.
- If the angles and one side length of a right-angled triangle are known then the other side lengths can be found using the sin θ, cos θ or tan θ ratios.



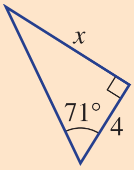
$$\sin 30^\circ = \frac{x}{3}$$

$$\therefore x = 3 \times \sin 30^\circ$$



$$\cos 42^\circ = \frac{x}{7.2}$$

$$\therefore x = 7.2 \times \cos 42^\circ$$



$$\tan 71^\circ = \frac{x}{4}$$

$$\therefore x = 4 \times \tan 71^\circ$$

Example 8 Using a calculator

Use a calculator to evaluate the following, correct to two decimal places.

a $\sin 50^\circ$ **b** $\cos 16^\circ$ **c** $\tan 77^\circ$

SOLUTION	EXPLANATION
a $\sin 50^\circ = 0.77$ (to 2 d.p.)	$\sin 50^\circ = 0.766044\ldots$ the 3rd decimal place is greater than 4 so round up.
b $\cos 16^\circ = 0.96$ (to 2 d.p.)	$\cos 16^\circ = 0.961261\ldots$ the 3rd decimal place is less than 5 so round down.
c $\tan 77^\circ = 4.33$ (to 2 d.p.)	$\tan 77^\circ = 4.331475\ldots$ the 3rd decimal place is less than 5 so round down.

Example 9 Solving for x in the numerator of a trigonometric ratio

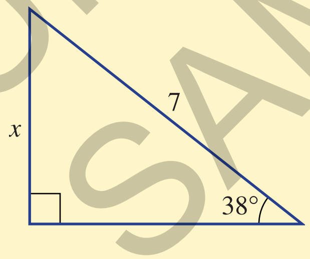
Find the value of x in the equation $\cos 20^\circ = \frac{x}{3}$, correct to two decimal places.

SOLUTION	EXPLANATION
$\cos 20^\circ = \frac{x}{3}$	
$x = 3 \times \cos 20^\circ$ $= 2.82$ (to 2 d.p.)	Multiply both sides of the equation by 3 and round as required.

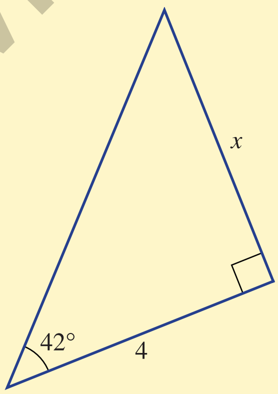
Example 10 Finding side lengths

For each triangle, find the value of x correct to two decimal places.

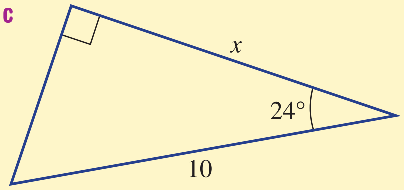
a

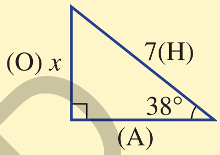
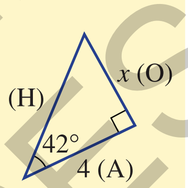
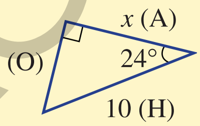


b



c



SOLUTION	EXPLANATION
a $\sin 38^\circ = \frac{O}{A}$ $\sin 38^\circ = \frac{x}{7}$ $x = 7\sin 38^\circ$ $= 4.31$ (to 2 d.p.)	Since the opposite side (O) and the hypotenuse (H) are involved, the sin θ ratio must be used. Multiply both sides by 7 and evaluate using a calculator. 
b $\tan 42^\circ = \frac{O}{A}$ $\tan 42^\circ = \frac{x}{4}$ $x = 4\tan 42^\circ$ $= 3.60$ (to 2 d.p.)	Since the opposite side (O) and the adjacent side (A) are involved, the tan θ ratio must be used. Multiply both sides by 4 and evaluate. 
c $\cos 24^\circ = \frac{A}{H}$ $\cos 24^\circ = \frac{x}{10}$ $x = 10\cos 24^\circ$ $= 9.14$ (to 2 d.p.)	Since the adjacent side (A) and the hypotenuse (H) are involved, the cos θ ratio must be used. Multiply both sides by 10. 

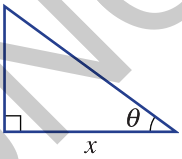
Exercise 3F

1–3	3(½)	—

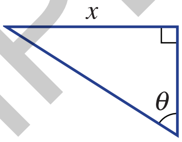
UNDERSTANDING

1 For the marked angle θ, decide if x represents the length of the opposite (O), adjacent (A) or hypotenuse (H) side.

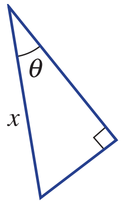
a



b

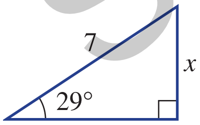


c



2 Decide if you would use $\sin\theta = \frac{O}{H}$, $\cos\theta = \frac{A}{H}$ or $\tan\theta = \frac{O}{A}$ to help find the value of x in these triangles. Do not find the value of x, just state which ratio would be used.

a



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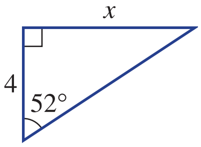
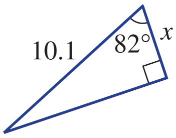


c



3F

UNDERSTANDING

Example 8

3 Use a calculator to evaluate the following correct to two decimal places.

a sin 20°	**b** cos 37°	**c** tan 64°	**d** sin 47°
e cos 84°	**f** tan 14.1°	**g** sin 27.4°	**h** cos 76.2°

4–5(½)	4–5(½)	4–5(½)

FLUENCY

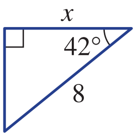
Example 9

4 In each of the following, find the value of x correct to two decimal places.

a $\sin 50^\circ = \frac{x}{4}$	**b** $\tan 81^\circ = \frac{x}{3}$	**c** $\cos 33^\circ = \frac{x}{6}$
d $\cos 75^\circ = \frac{x}{3.5}$	**e** $\sin 24^\circ = \frac{x}{4.2}$	**f** $\tan 42^\circ = \frac{x}{10}$
g $\frac{x}{7.1} = \tan 18.4^\circ$	**h** $\frac{x}{5.3} = \sin 64.7^\circ$	**i** $\frac{x}{12.6} = \cos 52.9^\circ$

Example 10

5 For the triangles given below, find the value of x correct to two decimal places.

a


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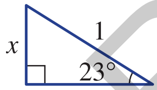
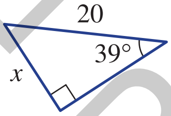
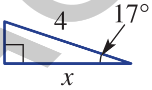
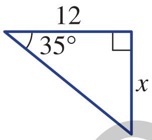
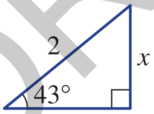
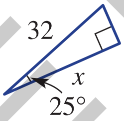


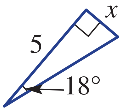
c


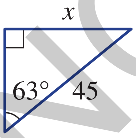
d


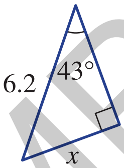
e


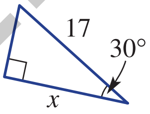
f


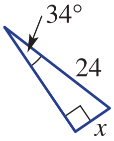
g


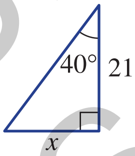
h


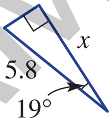
i


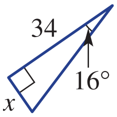
j


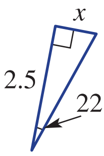
k


l


m


n


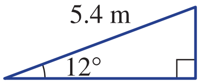
o


p


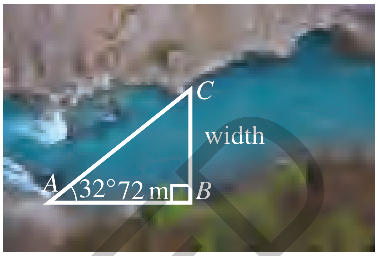
6, 7	6–8	7–9

PROBLEM-SOLVING

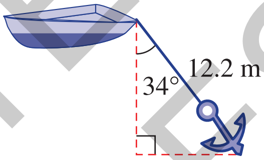
6 Amy walks 5.4 m up a ramp which is inclined at 12° to the horizontal. How high (correct to two decimal places) is she above her starting point?



7 Kane wanted to measure the width of a river. He placed two markers, A and B, 72 m apart along the bank. C is a point directly opposite marker B. Kane measured angle CAB to be 32°. Find the width of the river correct to two decimal places.

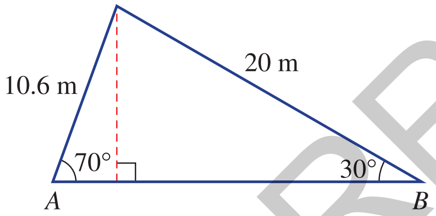


8 One end of a 12.2 m rope is tied to a boat. The other end is tied to an anchor, which is holding the boat steady in the water. If the anchor is making an angle of 34° with the vertical, how deep is the water? Give your answer correct to two decimal places.

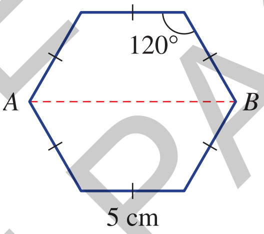


9 Find the length AB in these diagrams. Round to two decimal places where necessary.

a



b



10	10	10, 11

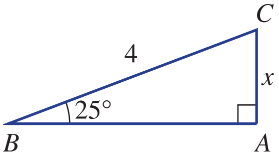
10 For this right-angled triangle:

a Find the value of $\angle C$.

b Calculate the value of x correct to three decimal places using the sine ratio.

c Calculate the value of x correct to three decimal places but instead use the cosine ratio.

d Comment on your answers to parts **b** and **c**.



11 Complementary angles sum to 90°.

a Find the complementary angles to these angles.

i 10° **ii** 28° **iii** 54° **iv** 81°

b Evaluate:

i sin 10° and cos 80° **ii** sin 28° and cos 62°

iii cos 54° and sin 36° **iv** cos 81° and sin 9°

c What do you notice in part **b**?

d Complete the following.

i sin 20° = cos _____ **ii** sin 59° = cos _____

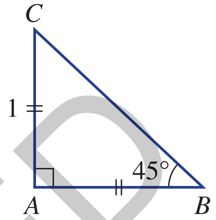
iii cos 36° = sin _____ **iv** cos 73° = sin _____

3F Exact values

— — 12

ENRICHMENT

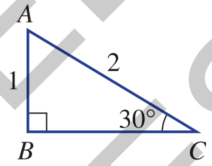
12 $\sqrt{2}, \sqrt{3}$ and $\dfrac{1}{\sqrt{2}}$ are examples of exact values.



a For the triangle shown (right), use Pythagoras' theorem to find the exact length *BC*.

b Use your result from part **a** to write down the exact values of:

i sin 45° **ii** cos 45° **iii** tan 45°



c For this triangle (right) use Pythagoras' theorem to find the exact length *BC*.

d Use your result from part **c** to write down the exact values of:

i sin 30° **ii** cos 30° **iii** tan 30°

iv sin 60° **v** cos 60° **vi** tan 60°

This diagram by the third century AD Chinese mathematician Liu Hui shows how to measure the height of a mountain on a sea island using right-angled triangles. This method of surveying became known as triangulation.

3G Solving for the denominator

Interactive

Widgets

HOTsheets

Walkthroughs

So far we have constructed trigonometric ratios using a pronumeral which has always appeared in the numerator.

For example: $\frac{x}{5} = \sin 40°$.

This makes it easy to solve for x where both sides of the equation can be multiplied by 5.

If, however, the pronumeral appears in the denominator there are a number of algebraic steps that can be taken to find the solution.

Let's start: Solution steps

Three students attempt to solve $\sin 40° = \frac{5}{x}$ for x.

Nick says $x = 5 \times \sin 40°$

Sharee says $x = \frac{5}{\sin 40°}$

Dori says $x = \frac{1}{5} \times \sin 40°$

- Which student has the correct solution?
- Can you show the algebraic steps that support the correct answer?

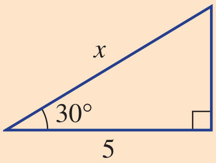
Key ideas

■ If the unknown value of a trigonometric ratio is in the **denominator**, you need to rearrange the equation to make the pronumeral the subject.

For example: For the triangle shown, $\cos 30° = \frac{5}{x}$

Multiplying both sides by x $\quad x \times \cos 30° = 5$

Dividing both sides by cos 30° $\quad x = \frac{5}{\cos 30°}$



Example 11 Solving for x in the denominator

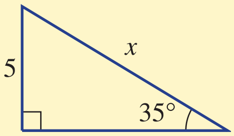
Solve for x in the equation $\cos 35° = \frac{2}{x}$, correct to two decimal places.

SOLUTION	EXPLANATION
$\cos 35° = \frac{2}{x}$	
$x \cos 35° = 2$	Multiply both sides of the equation by x.
$x = \frac{2}{\cos 35°}$	Divide both sides of the equation by cos 35°.
$= 2.44$ (to 2 d.p.)	Evaluate and round to two decimal places.

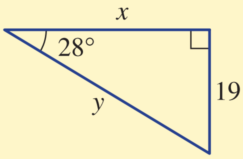
Example 12 Finding side lengths

Find the values of the pronumerals correct to two decimal places.

a



b



SOLUTION

a

$$\sin 35^\circ = \frac{\text{O}}{\text{H}}$$

$$\sin 35^\circ = \frac{5}{x}$$

$$x \sin 35^\circ = 5$$

$$x = \frac{5}{\sin 35^\circ}$$

$$= 8.72 \text{ (to 2 d.p.)}$$

b

$$\tan 28^\circ = \frac{\text{O}}{\text{A}}$$

$$\tan 28^\circ = \frac{19}{x}$$

$$x \tan 28^\circ = 19$$

$$x = \frac{19}{\tan 28^\circ}$$

$$= 35.73$$

$$y^2 = x^2 + 19^2$$

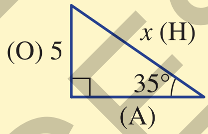
$$= 1637.904\ldots$$

$$y = \sqrt{1637.904\ldots}$$

$$\therefore y = 40.47 \text{ (to 2 d.p.)}$$

EXPLANATION

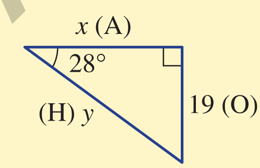
Since the opposite side (O) is given and we require the hypotenuse (H), use sin θ.



Multiply both sides of the equation by x then divide both sides of the equation by sin 35°.

Evaluate on a calculator and round to two decimal places.

Since the opposite side (O) is given and the adjacent (A) is required, use tan θ.

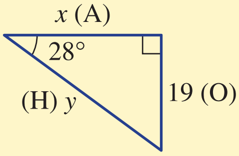


Multiply both sides of the equation by x.

Divide both sides of the equation by tan 28° and round the answer to two decimal places.

Find y by using Pythagoras' theorem and substitute the exact value of x, i.e. $\frac{19}{\tan 28^\circ}$

Alternatively, y can be found by using sin θ.



Exercise 3G

1–2(½) 2(½) —

UNDERSTANDING

1 Solve these simple equations for x.

a $\frac{4}{x} = 2$

b $\frac{20}{x} = 4$

c $\frac{15}{x} = 5$

d $25 = \frac{100}{x}$

e $5 = \frac{35}{x}$

f $\frac{10}{x} = 2.5$

g $\frac{2.5}{x} = 5$

h $12 = \frac{2.4}{x}$

UNDERSTANDING

Example 11

2 For each of the following equations, find the value of x correct to two decimal places.

a $\cos 43^\circ = \frac{3}{x}$

b $\sin 36^\circ = \frac{4}{x}$

c $\tan 9^\circ = \frac{6}{x}$

d $\tan 64^\circ = \frac{2}{x}$

e $\cos 67^\circ = \frac{5}{x}$

f $\sin 12^\circ = \frac{3}{x}$

g $\sin 38.3^\circ = \frac{5.9}{x}$

h $\frac{45}{x} = \tan 21.4^\circ$

i $\frac{18.7}{x} = \cos 32^\circ$

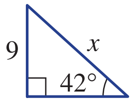
3–4(½)	3–4(½)	3–4(½)

FLUENCY

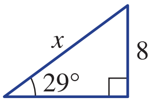
Example 12a

3 Find the value of x correct to two decimal places using the sine, cosine or tangent ratios.

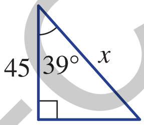
a



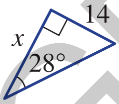
b



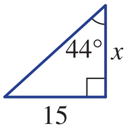
c



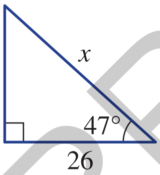
d



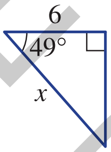
e



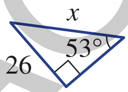
f



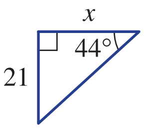
g



h



i



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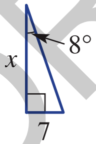

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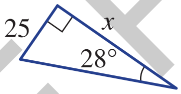

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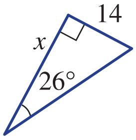
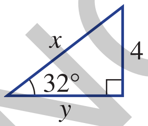


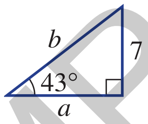
Example 12b

4 Find the value of each pronumeral correct to one decimal place.

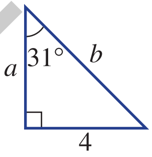
a



b



c



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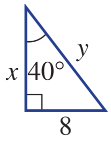
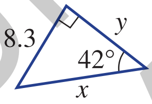


e



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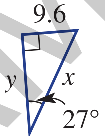

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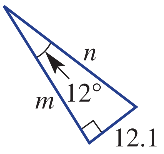
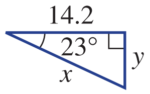


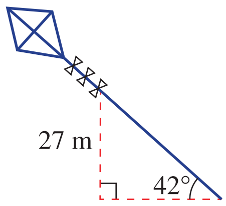
h



5, 6	5, 6, 7a	6, 7

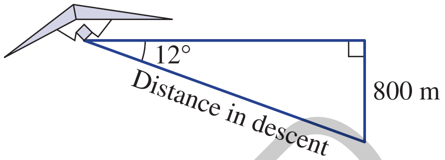
PROBLEM-SOLVING

5 A kite is flying at a height of 27 m above the anchor point. If the string is inclined at 42° to the horizontal, find the length of the string, correct to the nearest metre.



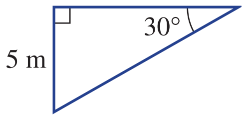
PROBLEM-SOLVING

6 A paraglider flying at a height of 800 m descends at an angle of 12° to the horizontal. How far (to the nearest metre) has it travelled in descending to the ground?

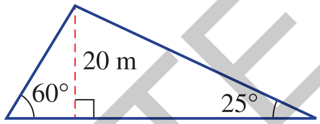


7 Find the perimeter of these triangles, correct to one decimal place.

a



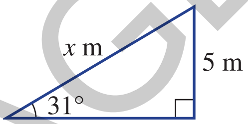
b



8	8	8

REASONING

8 In calculating the value of x for this triangle, correct to two decimal places, two students come up with these answers.



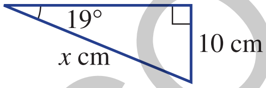
A $x = \dfrac{5}{\sin 31°} = \dfrac{5}{0.52} = 9.62$ **B** $x = \dfrac{5}{\sin 31°} = 9.71$

a Which of the above two answers is more correct and why?

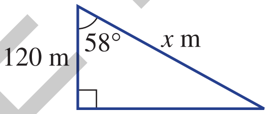
b What advice would you give to the student whose answer is not accurate?

c Find the difference in the answers if the different methods (A and B) are used to calculate the value of x correct to two decimal places in these triangles.

i



ii



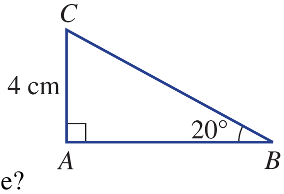
Linking tan θ to sin θ and cos θ

—	—	9

ENRICHMENT

9 **a** For this triangle find, correct to three decimal places:

i AB **ii** BC



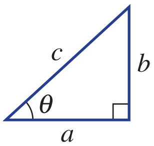
b Calculate these ratios to two decimal places.

i sin 20° **ii** cos 20° **iii** tan 20°

c Evaluate $\dfrac{\sin 20°}{\cos 20°}$ using your results from part **b**. What do you notice?

d For this triangle with side lengths a, b and c, find an expression for:

i $\sin\,\theta$ **ii** $\cos\,\theta$ **iii** $\tan\,\theta$ **iv** $\dfrac{\sin\,\theta}{\cos\,\theta}$



e Simplify your expression for part **d iv**. What do you notice?

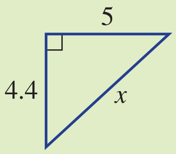
Using a CAS calculator 3G: Trigonometry

This activity is in the interactive textbook in the form of a printable PDF.

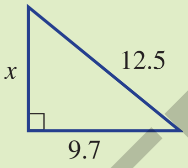
Progress quiz

3A/B **1** Find the length of the missing side in these right-angled triangles. Round to two decimal places.

a

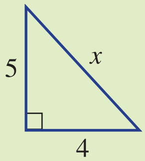


b

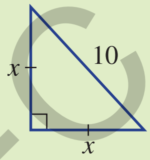


3A/B **2** Find the exact value of x in these right-angled triangles.

a



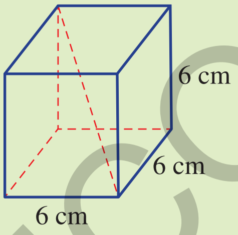
b



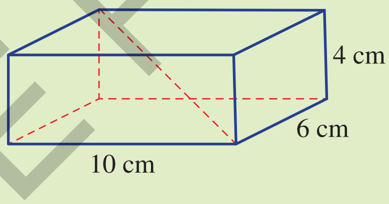
3C **3** A ladder 230 cm long is placed 50 cm from the edge of a building, how far up the side of the building will this ladder reach? Round to one decimal place.

3D **Ext** **4** Find the length of the diagonals of these prisms, correct to one decimal place.

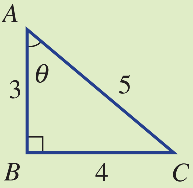
a



b



3E **5** Consider the triangle ABC.



a Name the hypotenuse.
b Name the side adjacent to angle ACB.
c Write the ratio for $\cos\ \theta$.
d Write the ratio for $\tan\ \theta$.

3F/G **6** Solve for x, correct to two decimal places.

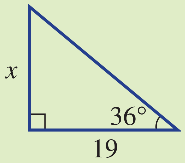
a $x = 12.7 \cos 54^\circ$

b $\tan 30^\circ = \frac{x}{12}$

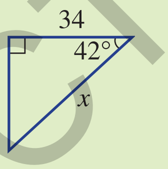
c $\sin 56^\circ = \frac{58.4}{x}$

3F/G **7** Find the values of the pronumerals, correct to two decimal places.

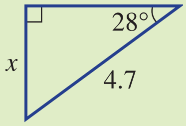
a



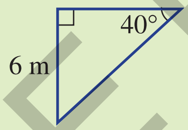
b



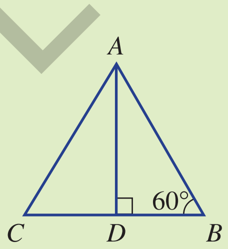
c



3G **8** Find the perimeter of this triangle, correct to two decimal places.



3A–G **9** Triangle ABC is equilateral with a perimeter of 12 cm.



Find:

a the height AD using any suitable method, correct to three decimal places

b the area of the triangle ABC, correct to one decimal place.

3H Finding an angle

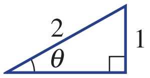
Interactive

Widgets

HOTsheets

Walkthroughs

Logically, if you can use trigonometry to find a side length of a right-angled triangle given one angle and one side, you should be able to find an angle if you are given two sides.



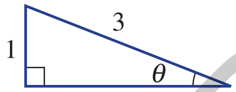
We know that $\sin 30° = \frac{1}{2}$ so if we were to determine θ if $\sin \theta = \frac{1}{2}$, the answer would be $\theta = 30°$.

We write this as $\theta = \sin^{-1}\left(\frac{1}{2}\right) = 30°$ and we say that the inverse sine of $\frac{1}{2}$ is 30°.

Calculators can be used to help solve problems using inverse sine ($\sin^{-1}$), inverse cosine ($\cos^{-1}$) and inverse tangent ($\tan^{-1}$). For angles in degrees, ensure your calculator is in degree mode.

Let's start: Trial and error can be slow

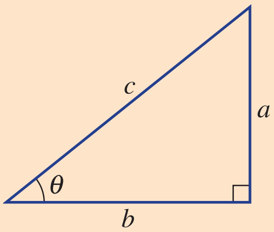
We know that for this triangle, $\sin \theta = \frac{1}{3}$.



- Guess the angle θ.
- For your guess use a calculator to see if $\sin \theta = \frac{1}{3} = 0.333\ldots$
- Update your guess and use your calculator to check once again.
- Repeat this trial-and-error process until you think you have the angle θ correct to three decimal places.
- Now evaluate $\sin^{-1}\left(\frac{1}{3}\right)$ and check your guess.

Key ideas

- **Inverse sine** ($\sin^{-1}$), **inverse cosine** ($\cos^{-1}$) and **inverse tangent** ($\tan^{-1}$) can be used to find angles in right-angled triangles.
  - $\sin \theta = \frac{a}{c}$ means $\theta = \sin^{-1}\left(\frac{a}{c}\right)$
  - $\cos \theta = \frac{b}{c}$ means $\theta = \cos^{-1}\left(\frac{b}{c}\right)$
  - $\tan \theta = \frac{a}{b}$ means $\theta = \tan^{-1}\left(\frac{a}{b}\right)$
- Note that $\sin^{-1} x$ does *not* mean $\frac{1}{\sin x}$.



Example 13 Using inverse trigonometric ratios

Find the value of θ to the level of accuracy indicated.

a $\sin\theta = 0.3907$ (nearest degree) **b** $\tan\theta = \frac{1}{2}$ (one decimal place)

SOLUTION

a $\sin\theta = 0.3907$

$\theta = \sin^{-1}(0.3907)$

$= 23^\circ$ (to nearest degree)

b $\tan\theta = \frac{1}{2}$

$\theta = \tan^{-1}\left(\frac{1}{2}\right)$

$= 26.6^\circ$ (to 1 d.p.)

EXPLANATION

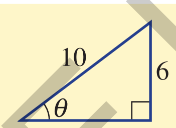
Use the $\sin^{-1}$ key on your calculator.

Round to the nearest whole number.

Use the $\tan^{-1}$ key on your calculator and round the answer to one decimal place.

Example 14 Finding an angle

Find the value of θ to the nearest degree.



SOLUTION

$\sin\theta = \frac{O}{H}$

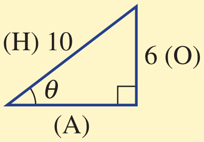
$= \frac{6}{10}$

$\theta = \sin^{-1}\left(\frac{6}{10}\right)$

$= 37^\circ$

EXPLANATION

Since the opposite side (O) and the hypotenuse (H) are given, use $\sin\theta$.



Use the $\sin^{-1}$ key on your calculator and round as required.

Exercise 3H

1, 2, 3(½), 4	4	—

UNDERSTANDING

1 Use a calculator to evaluate the following rounding to two decimal places.

a $\sin^{-1}(0.2)$ **b** $\sin^{-1}(0.9)$ **c** $\cos^{-1}(0.75)$

d $\cos^{-1}(0.43)$ **e** $\tan^{-1}(0.5)$ **f** $\tan^{-1}(2.5)$

UNDERSTANDING

2 Write the missing number.

a If $\sin 30^\circ = \frac{1}{2}$ then $30^\circ = \sin^{-1}($ _____ $)$.

b If $\cos 50^\circ = 0.64$ then _____ $= \cos^{-1}(0.64)$.

c If $\tan 45^\circ = 1$ then _____ $= \tan^{-1}($ _____ $)$.

3 Evaluate each of the following to the nearest degree.

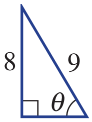
a $\sin^{-1}(0.7324)$ **b** $\cos^{-1}(0.9763)$ **c** $\tan^{-1}(0.3321)$

d $\tan^{-1}(1.235)$ **e** $\sin^{-1}(0.4126)$ **f** $\cos^{-1}(0.7462)$

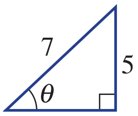
g $\cos^{-1}(0.1971)$ **h** $\sin^{-1}(0.2247)$ **i** $\tan^{-1}(0.0541)$

4 Which trigonometric ratio should be used to solve for θ?

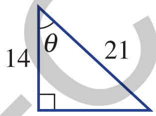
a



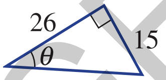
b



c



d



5–6(½), 7	5–7(½)	5–7(½)

FLUENCY

Example 13a

5 Find the value of θ to the nearest degree.

a $\sin \theta = 0.5$ **b** $\cos \theta = 0.5$ **c** $\tan \theta = 1$

d $\cos \theta = 0.8660$ **e** $\sin \theta = 0.7071$ **f** $\tan \theta = 0.5774$

g $\sin \theta = 1$ **h** $\tan \theta = 1.192$ **i** $\cos \theta = 0$

j $\cos \theta = 0.5736$ **k** $\cos \theta = 1$ **l** $\sin \theta = 0.9397$

Example 13b

6 Find the angle θ correct to two decimal places.

a $\sin \theta = \frac{4}{7}$ **b** $\sin \theta = \frac{1}{3}$ **c** $\sin \theta = \frac{9}{10}$

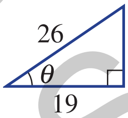
d $\cos \theta = \frac{1}{4}$ **e** $\cos \theta = \frac{4}{5}$ **f** $\cos \theta = \frac{7}{9}$

g $\tan \theta = \frac{3}{5}$ **h** $\tan \theta = \frac{8}{5}$ **i** $\tan \theta = 12$

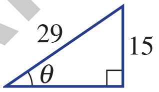
Example 14

7 Find the value of θ to the nearest degree.

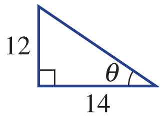
a



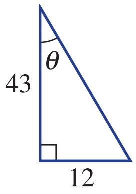
b



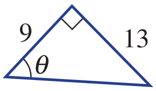
c



d



e



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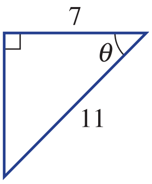
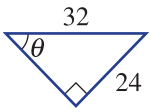
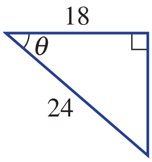


g



h

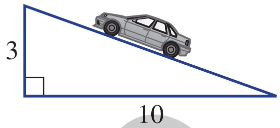


3H

8–10	9–11	10–12

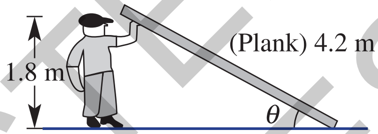
PROBLEM-SOLVING

8 A road rises at a grade of 3 in 10. Find the angle (to the nearest degree) the road makes with the horizontal.

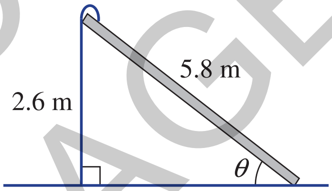


9 When a 2.8 m long seesaw is at its maximum height it is 1.1 m off the ground. What angle (correct to two decimal places) does the seesaw make with the ground?

10 Adam, who is 1.8 m tall, holds up a plank of wood 4.2 m long. Find the angle that the plank makes with the ground, correct to one decimal place.

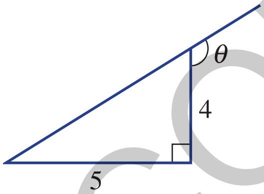


11 A children's slide has a length of 5.8 m. The vertical ladder is 2.6 m above the ground. Find the angle the slide makes with the ground, correct to one decimal place.

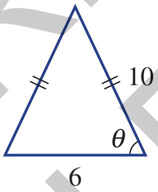


12 Find the value of θ in these diagrams, correct to one decimal place.

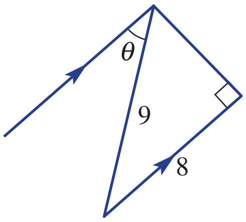
a



b



c



13	13, 14	14, 15

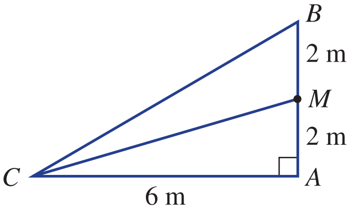
REASONING

13 Find all the angles to the nearest degree in right-angled triangles with these side lengths.

a 3, 4, 5 **b** 5, 12, 13 **c** 7, 24, 25

14 For what value of θ is $\sin \theta = \cos \theta$?

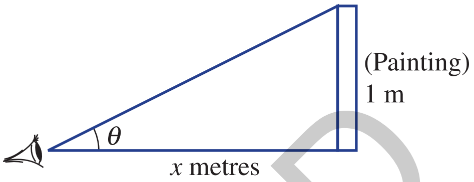
15 If M is the midpoint of AB, decide if $\angle ACM$ is exactly half of angle $\angle ACB$. Investigate and explain.



Viewing angle

—	—	16

16 Jo has forgotten her glasses and is trying to view a painting in a gallery. Her eye level is at the same level as the base of the painting and the painting is 1 metre tall.
Answer the following to the nearest degree for angles and to two decimal places for lengths.



- **a** If $x = 3$, find the viewing angle θ.
- **b** If $x = 2$, find the viewing angle θ.
- **c** If Jo can stand no closer than 1 metre to the painting, what is Jo's largest viewing angle?
- **d** When the viewing angle is 10°, Jo has trouble seeing the painting. How far is she from the painting at this viewing angle?

  Theoretically, what would be the largest viewing angle if Jo could go as close as she would like to the painting?

3I Applying trigonometry EXTENDING

Interactive
Widgets
HOTsheets
Walkthroughs

In many situations, angles are measured up or down from the horizontal. These are called angles of elevation and depression. Combined with the mathematics of trigonometry, these angles can be used to solve problems, provided right-angled triangles can be identified. The line of sight to a helicopter 100 m above the ground, for example, creates an angle of elevation inside a right-angled triangle.

Let's start: Illustrate the situation

For the situation below, draw a detailed diagram showing these features:

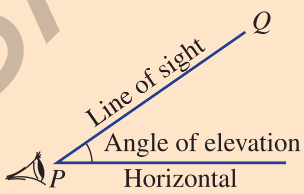
- an angle of elevation
- an angle of depression
- any given lengths
- a right-angled triangle that will help to solve the problem

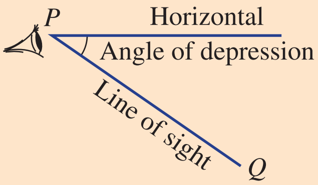
A cat and a bird eye each other from their respective positions. The bird is 20 m up a tree and the cat is on the ground 30 m from the base of the tree. Find the angle their line of sight makes with the horizontal.

Compare your diagram with others in your class. Is there more than one triangle that could be drawn and used to solve the problem?

Key ideas

- To solve application problems involving trigonometry:
  1. Draw a diagram and label the key information.
  2. Identify and draw the appropriate right-angled triangles separately.
  3. Solve using trigonometry to find the missing measurements.
  4. Express your answer in words.
- The **angle of elevation** or **depression** of a point, Q, from another point, P, is given by the angle the line PQ makes with the horizontal.

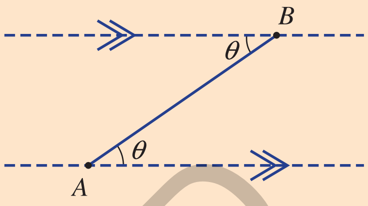




- Angles of elevation or depression are always measured from the horizontal.

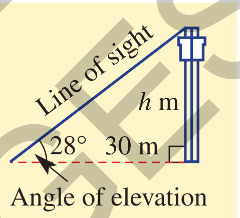
Key ideas

- In this diagram the angle of elevation of *B* from *A* is equal to the angle of depression of *A* from *B*. They are equal alternate angles in parallel lines.



Example 15 Using angles of elevation

The angle of elevation of the top of a tower from a point on the ground 30 m away from the base of the tower is 28°. Find the height of the tower to the nearest metre.



SOLUTION

Let the height of the tower be *h* m.

$$\tan 28^\circ = \frac{\text{O}}{\text{A}}$$

$$= \frac{h}{30}$$

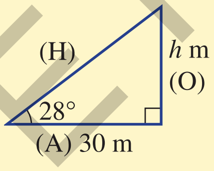
$$h = 30 \tan 28^\circ$$

$$= 15.951\ldots$$

The height is 16 m, to the nearest metre.

EXPLANATION

Since the opposite side (O) is required and the adjacent (A) is given, use tan θ.

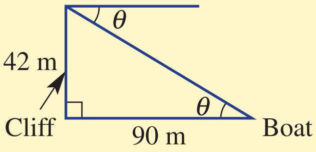


Multiply both sides by 30 and evaluate. Round to the nearest metre and write the answer in words.

Example 16 Finding an angle of depression

From the top of a vertical cliff Andrea spots a boat out at sea. If the top of the cliff is 42 m above sea level and the boat is 90 m away from the base of the cliff, find Andrea's angle of depression to the boat to the nearest degree.

SOLUTION



EXPLANATION

Draw a diagram and label all the given measurements.

Use alternate angles in parallel lines to mark θ inside the triangle.

$$\tan\ \theta = \frac{\text{O}}{\text{A}}$$

$$= \frac{42}{90}$$

$$\theta = \tan^{-1}\left(\frac{42}{90}\right)$$

$$\theta = 25.0168\ldots^\circ$$

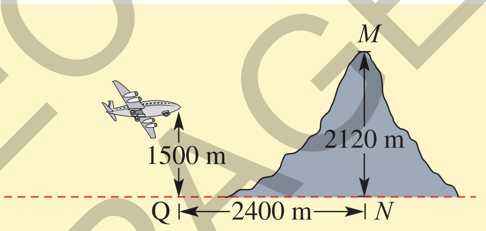
The angle of depression is 25°, to the nearest degree.

Since the opposite (O) and adjacent sides (A) are given, use tan θ.

Use the $\tan^{-1}$ key on your calculator. Round to the nearest degree and express the answer in words.

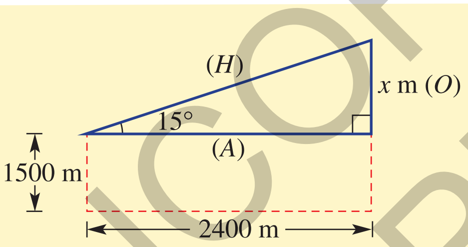
Example 17 Applying trigonometry

A plane flying at an altitude of 1500 m starts to climb at an angle of 15° to the horizontal when the pilot sees a mountain peak 2120 m high, 2400 m away from him horizontally. Will the pilot clear the mountain?



SOLUTION

EXPLANATION



Draw a diagram, identifying and labelling the right-angled triangle to help solve the problem. The plane will clear the mountain if the opposite (O) is greater than

(2120 – 1500) m = 620 m.

$$\tan 15^\circ = \frac{x}{2400}$$

$$x = 2400 \tan 15^\circ$$

$$= 643.078\ldots$$

x needs to be greater than
2120 – 1500 = 620

Since $x > 620$ the plane will clear the mountain peak.

Set up the trigonometric ratio using tan.

Multiply by 2400 and evaluate.

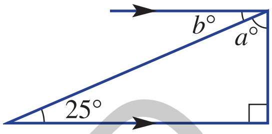
Answer the question in words.

Exercise 3I

1–3	3	—

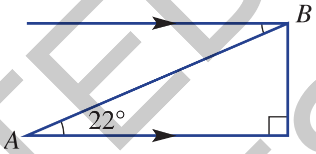
UNDERSTANDING

1 Find the values of the pronumerals in this diagram.



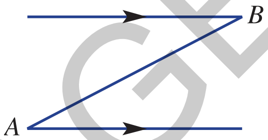
2 For this diagram:

- a what is the angle of elevation of *B* from *A*?
- b what is the angle of depression of *A* from *B*?



3 Draw this diagram and complete these tasks.

- a Mark in the following:
  - i the angle of elevation (θ) of *B* from *A*
  - ii the angle of depression (α) of *A* from *B*.
- b Is $\theta = \alpha$ in your diagram? Why?

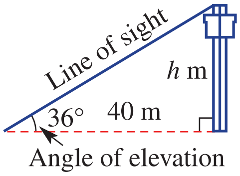


4–9	4–6, 8–10	4, 6, 8–11

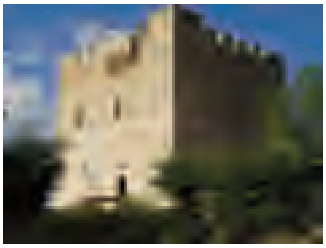
FLUENCY

Example 15

4 The angle of elevation of the top of a tower from a point on the ground 40 m from the base of the tower is 36°. Find the height of the tower to the nearest metre.

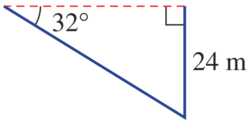


5 The angle of elevation of the top of a castle wall from a point on the ground 25 m from the base of the castle wall is 32°. Find the height of the castle wall to the nearest metre.



6 From a point on the ground, Emma measures the angle of elevation of an 80 m tower to be 27°. Find how far Emma is from the base of the tower, correct to the nearest metre.

7 From a pedestrian overpass, Chris spots a landmark at an angle of depression of 32°. How far away (to the nearest metre) is the landmark from the base of the 24 m high overpass?



8 From a lookout tower, David spots a bushfire at an angle of depression of 25°. If the lookout tower is 42 m high, how far away (to the nearest metre) is the bushfire from the base of the tower?

FLUENCY

Example 16

9 From the top of a vertical cliff, Josh spots a swimmer out at sea. If the top of the cliff is 38 m above sea level and the swimmer is 50 m away from the base of the cliff, find the angle of depression from Josh to the swimmer, to the nearest degree.

10 From a ship, a person is spotted floating in the sea 200 m away. If the viewing position on the ship is 20 m above sea level, find the angle of depression from the ship to person in the sea. Give your answer to the nearest degree.

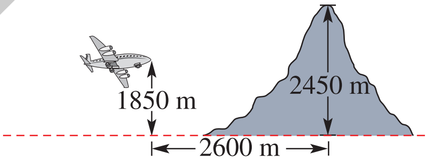
11 A power line is stretched from a pole to the top of a house. The house is 4.1 m high and the power pole is 6.2 m high. The horizontal distance between the house and the power pole is 12 m. Find the angle of elevation of the top of the power pole from the top of the house, to the nearest degree.

12, 13	12–14	12, 15, 16

PROBLEM-SOLVING

Example 17

12 A plane flying at 1850 m starts to climb at an angle of 18° to the horizontal when the pilot sees a mountain peak 2450 m high, 2600 m away from him in a horizontal direction. Will the pilot clear the mountain?

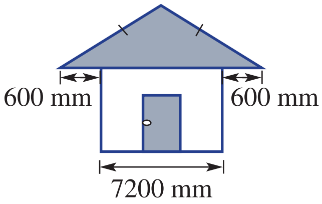


13 A road has a steady gradient of 1 in 10.

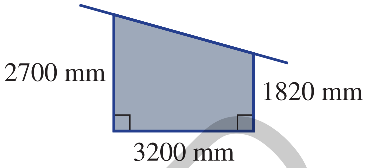
a What angle does the road make with the horizontal? Give your answer to the nearest degree.

b A car starts from the bottom of the inclined road and drives 2 km along the road. How high vertically has the car climbed? Use your rounded answer from part **a** and give your answer correct to the nearest metre.

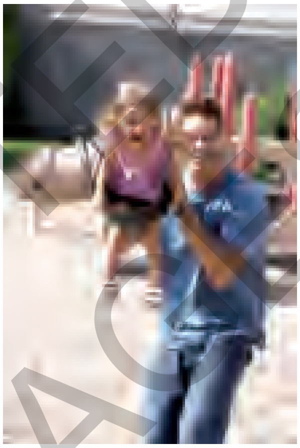
14 A house is to be built using the design shown on the right. The eaves are 600 mm and the house is 7200 mm wide, excluding the eaves. Calculate the length (to the nearest mm) of a sloping edge of the roof, which is pitched at 25° to the horizontal.



15 A garage is to be built with measurements as shown in the diagram on the right. Calculate the sloping length and pitch (angle) of the roof if the eaves extend 500 mm on each side. Give your answers correct to the nearest unit.

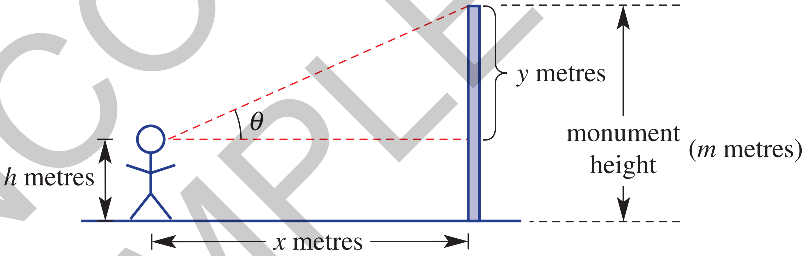


16 The chains on a swing are 3.2 m long and the seat is 0.5 m off the ground when it is in the vertical position. When the swing is pulled as far back as possible, the chains make an angle of 40° with the vertical. How high off the ground, to the nearest cm, is the seat when it is at this extreme position?



17	17	17, 18

17 A person views a vertical monument x metres away as shown.



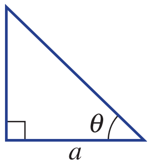
a If $h = 1.5$, $x = 20$ and $\theta = 15°$ find the height of the monument to two decimal places.

b If $h = 1.5$, $x = 20$ and $y = 10$ find θ correct to one decimal place.

c Let the height of the monument be m metres. Write expressions for the following:

- **i** m using (in terms of) y and h.
- **ii** y using x and θ.
- **iii** m using (in terms of) x, θ and h.

18 Find an expression for the area of this triangle using a and θ.



19 An aeroplane takes off and climbs at an angle of 20° to the horizontal, at 190 km/h along its flight path for 15 minutes.

a Find:

i the distance the aeroplane travels in 15 minutes

ii the height the aeroplane reaches after 15 minutes correct to two decimal places.

b If the angle at which the plane climbs is twice the original angle but its speed is halved will it reach a greater height after 15 minutes? Explain.

c If the plane's speed is doubled and its climbing angle is halved, will the plane reach a greater height after 15 minutes? Explain.

20 The residents of Skeville live 12 km from an airport. They maintain that any plane flying lower than 4 km disturbs their peace. Each Sunday they have an outdoor concert from 12.00 noon till 2.00 pm.

a Will a plane taking off from the airport at an angle of 15° over Skeville disturb the residents?

b When the plane in part **a** is directly above Skeville, how far (to the nearest metre) has it flown?

c If the plane leaves the airport at 11:50 am on Sunday and travels at an average speed of 180 km/h, will it disturb the start of the concert?

d Investigate what average speed (correct to the nearest km/h) the plane can travel at so that it does not disturb the concert. Assume it leaves at 11:50 am.

21 Peter observes a plane flying directly overhead at a height of 820 m. Twenty seconds later, the angle of elevation of the plane from Peter is 32°. Assume the plane flies horizontally.

a How far (to the nearest metre) did the plane fly in 20 seconds?

b What is the plane's speed in km/h, correct to the nearest km/h?

3J Bearings EXTENDING

Interactive

Widgets

HOTsheets

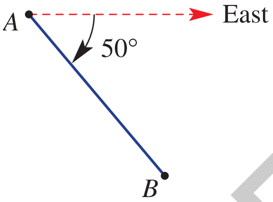
Walkthroughs

Bearings are used to indicate direction and therefore are commonly used to navigate the sea or air in ships or planes. Bushwalkers use bearings with a compass to help follow a map and navigate a forest. The most common type of bearing is the True bearing measured clockwise from north.


A compass can determine direction using Earth's magnetic field.

Let's start: Opposite directions

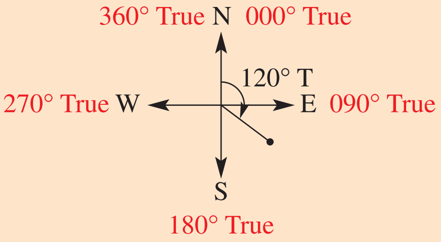
Marg at point *A* and Jim at point *B* start walking toward each other. Marg knows that she has to face 50° south of due east.



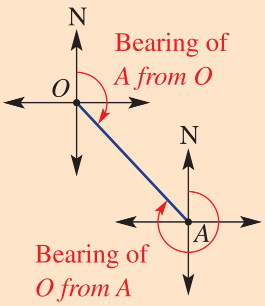
- Measured clockwise from north, can you help Marg determine her True compass bearing that she should walk on?
- Can you find what bearing Jim should walk on?
- Draw a detailed diagram which supports your answers above.

Key ideas

- A **True bearing** is an angle measured clockwise from north.
  - It is written using three digits.
    For example: 008° T, 032° T or 144° T.



- To describe the true bearing of an object positioned at *A* from an object positioned at *O*, we need to start at *O*, face north then turn clockwise through the required angle to face the object at *A*.

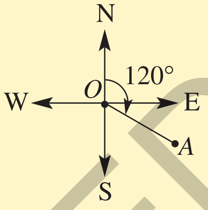


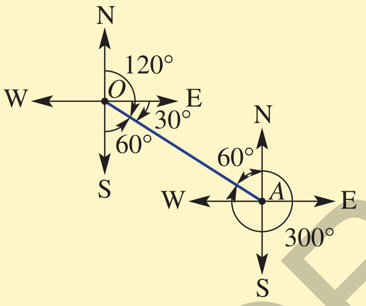
- When solving problems with bearings, draw a diagram including four point compass directions (N, E, S, W) at each point.

Example 18 Stating true bearings

For the diagram shown give:

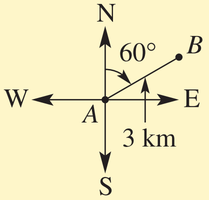
- **a** the true bearing of *A* from *O*
- **b** the true bearing of *O* from *A*.

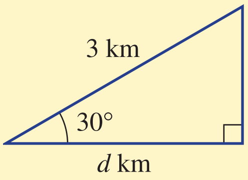


SOLUTION	EXPLANATION
a The bearing of *A* from *O* is 120° T.	Start at *O*, face north and turn clockwise until you are facing *A*.
b 	Start at *A*, face north and turn clockwise until you are facing *O*. Mark in a compass at *A* and use alternate angles in parallel lines to mark a 60° angle.
The bearing of *O* from *A* is: (360 – 60)° T = 300° T	True bearing is then 60° short of 360°.

Example 19 Using bearings with trigonometry

A bushwalker walks 3 km on a true bearing of 060° from point *A* to point *B*. Find how far east (correct to one decimal place) point *B* is from point *A*.



SOLUTION	EXPLANATION
Let the distance travelled towards the east be *d* km. 	Define the distance required and draw and label the right-angled triangle. Since the adjacent (A) is required and the hypotenuse (H) is given, use cos θ.

$$\cos 30° = \frac{d}{3}$$

$$d = 3\cos 30°$$

$$= 2.6 \text{ (to 1 d.p.)}$$

Multiply both sides of the equation by 3 and evaluate, rounding to one decimal place.

∴ The distance east is 2.6 km.

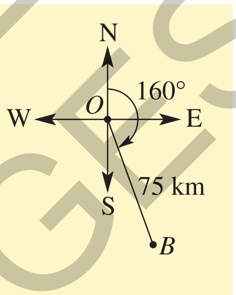
Express the answer in words.

Example 20 Calculating a bearing

A fishing boat starts from point *O* and sails 75 km on a bearing of 160° T to point *B*.

a How far east (to the nearest kilometre) of its starting point is the boat?

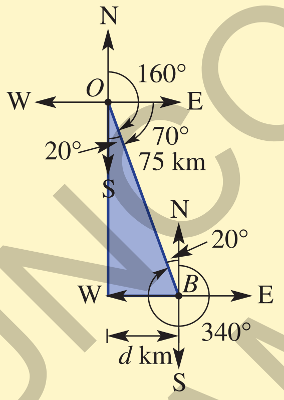
b What is the bearing of *O* from *B*?



SOLUTION

EXPLANATION

a Let the distance travelled towards the east be *d* km.



$$\sin 20° = \frac{d}{75}$$

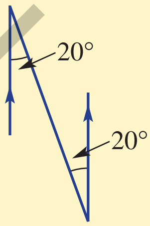
$$d = 75\sin 20°$$

$$= 25.651\ldots$$

Draw a diagram and label all the given measurements. Mark in a compass at *B* and use alternate angles to label extra angles.

Set up a trigonometric ratio using sine and solve for *d*.

Alternate angle = 20°



The boat has travelled 26 km to the east of its starting point, to the nearest kilometre.

Round to the nearest kilometre and write the answer in words.

b The bearing of *O* from *B* is (360 – 20)° T = 340° T

Start at *B*, face north then turn clockwise to face *O*.

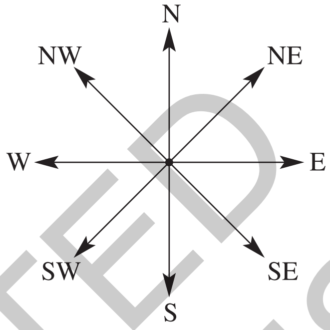
Exercise 3J

1, 2 | 2 | —

UNDERSTANDING

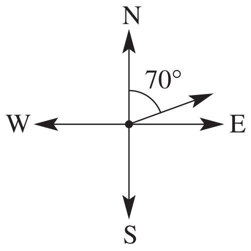
1 Give the true bearings for these common directions.

- **a** North (N)
- **b** North-east (NE)
- **c** East (E)
- **d** South-east (SE)
- **e** South (S)
- **f** South-west (SW)
- **g** West (W)
- **h** North-west (NW)

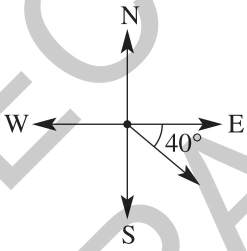


2 Write down the true bearings shown in these diagrams. Use three digits, for example, 045° T.

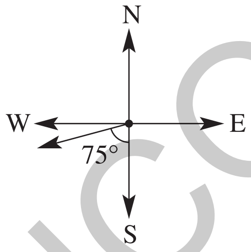
a



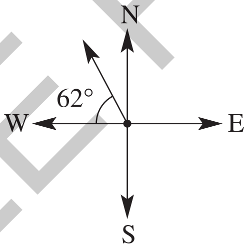
b



c



d



3(½), 4–6 | 3(½), 4–7 | 3(½), 4, 6, 7

FLUENCY

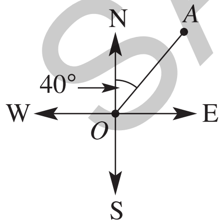
Example 18

3 For each diagram shown, write:

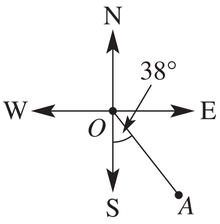
i the true bearing of *A* from *O*

ii the true bearing of *O* from *A*.

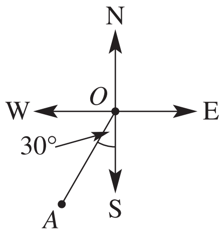
a



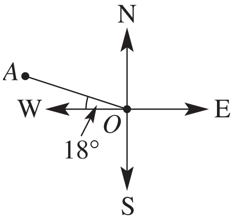
b



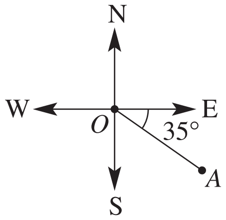
c



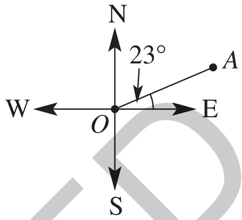
d



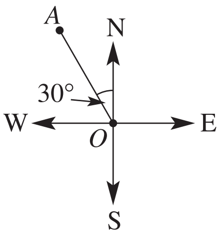
e



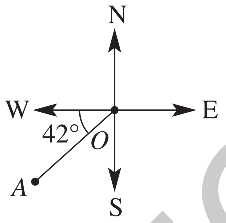
f



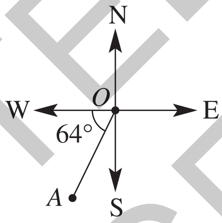
g



h

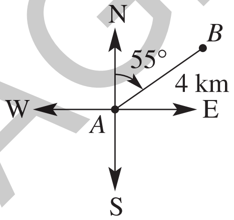


i

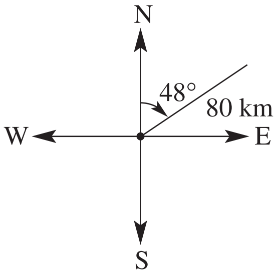


Example 19

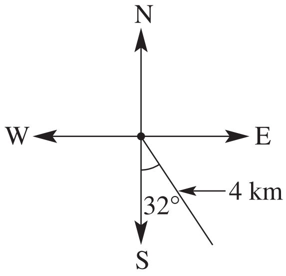
4 A bushwalker walks 4 km on a true bearing of 055° from point *A* to point *B*. Find how far east point *B* is from point *A*, correct to two decimal places.



5 A speed boat travels 80 km on a true bearing of 048°. Find how far east of its starting point the speed boat is, correct to two decimal places.



6 After walking due east, then turning and walking due south, a hiker is 4 km 148° T from her starting point. Find how far she walked in a southerly direction, correct to one decimal place.



7 A four-wheel drive vehicle travels for 32 km on a true bearing of 200°. How far west (to the nearest kilometre) of its starting point is it?

8–10	8–11	9–12

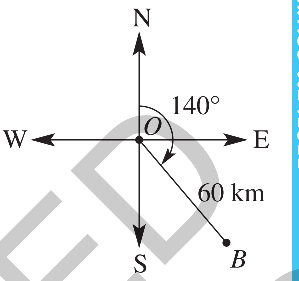
PROBLEM-SOLVING

Example 20

8 A fishing boat starts from point *O* and sails 60 km on a true bearing of 140° to point *B*.

a How far east of its starting point is the boat, to the nearest kilometre?

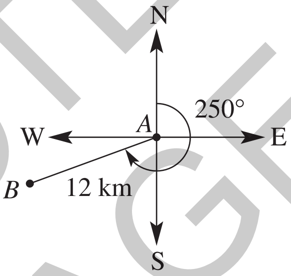
b What is the bearing of *O* from *B*?



9 Two towns, *A* and *B*, are 12 km apart. The true bearing of *B* from *A* is 250°.

a How far west of *A* is *B*, correct to one decimal place?

b Find the bearing of *A* from *B*.



10 A helicopter flies on a true bearing of 140° for 210 km then flies due east for 175 km. How far east (to the nearest kilometre) has the helicopter travelled from its starting point?

11 Christopher walks 5 km south then walks on a true bearing of 036° until he is due east of his starting point. How far is he from his starting point, to one decimal place?

12 Two cyclists leave from the same starting point. One cyclist travels due west while the other travels on a true bearing of 202°. After travelling for 18 km, the second cyclist is due south of the first cyclist. How far (to the nearest metre) has the first cyclist travelled?

13	13	13, 14

REASONING

13 A true bearing is *a*°. Write an expression for the true bearing of the opposite direction of *a*°if

a *a* is between 0 and 180

b *a* is between 180 and 360.

14 A hiker walks on a triangular pathway starting at point *A*, walking to point *B* then *C*, then *A* again as shown.

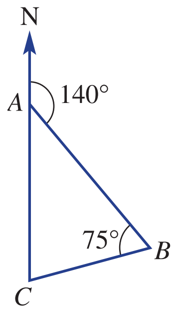
a Find the bearing from *B* to *A*.

b Find the bearing from *B* to *C*.

c Find the bearing from *C* to *B*.

d If the initial bearing was instead 133° and ∠*ABC* is still 75°, find the bearing from *B* to *C*.

e If ∠*ABC* was 42°, with the initial bearing of 140°, find the bearing from *B* to *C*.



Speed trigonometry — — 15, 16

ENRICHMENT

15 A plane flies on a true bearing of 168° for two hours at an average speed of 310 km/h. How far (to the nearest kilometre) has the plane travelled?

a south of its starting point is the plane?

b east of its starting point is the plane?

16 A pilot intends to fly directly to Anderly, which is 240 km due north of his starting point. The trip usually takes 50 minutes. Due to a storm, the pilot changes course and flies to Boxleigh on a true bearing of 320° for 150 km, at an average speed of 180 km/h.

a Find (to the nearest kilometre) how far:

- **i** north the plane has travelled from its starting point
- **ii** west the plane has travelled from its starting point.

b How many kilometres is the plane from Anderly?

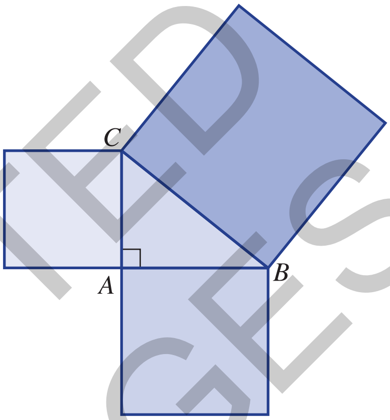
c From Boxleigh the pilot flies directly to Anderly at 240 km/h.

- **i** Compared to the usual route, how many extra kilometres (to the nearest kilometre) has the pilot travelled in reaching Anderly?
- **ii** Compared to the usual trip, how many extra minutes (correct to one decimal place) did the trip to Anderly take?

Investigation

Illustrating Pythagoras

It is possible to use a computer geometry package ('Cabri Geometry' or 'Geometers Sketchpad') to build this construction, which will illustrate Pythagoras' theorem.



Construct

a Start by constructing the line segment *AB*.

b Construct the right-angled triangle *ABC* by using the 'Perpendicular Line' tool.

c Construct a square on each side of the triangle. Circles may help to ensure your construction is exact.

Calculate

a Measure the areas of the squares representing AB^2, AC^2 and BC^2.

b Calculate the sum of the areas of the two smaller squares by using the 'Calculate' tool.

c **i** Drag point *A* or point *B* and observe the changes in the areas of the squares.

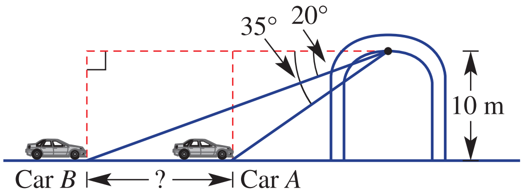
  ii Investigate how the areas of the squares change as you drag point *A* or point *B*. Explain how this illustrates Pythagoras' theorem.

Constructing triangles to solve problems

Illustrations for some problems may not initially look as if they include right-angled triangles. A common mathematical problem-solving technique is to construct right-angled triangles so that trigonometry can be used.

Car gap

Two cars are observed in the same lane from an overpass bridge 10 m above the road. The angles of depression to the cars are 20° and 35°.



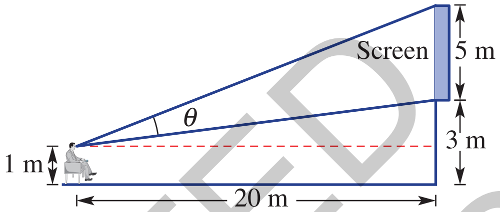
a Find the horizontal distance from car A to the overpass. Show your diagrams and working.

b Find the horizontal distance from car B to the overpass.

c Find the distance between the fronts of the two cars.

Cinema screen

A 5 m vertical cinema screen sits 3 m above the floor of the hall and Wally sits 20 m back from the screen. His eye level is 1 m above the floor.



a Find the angle of elevation from Wally's eye level to the base of the screen. Illustrate your method using a diagram.

b Find the angle of elevation as in part **a** but from his eye level to the top of the screen.

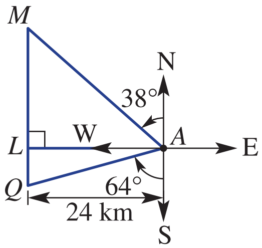
c Use your results from parts **a** and **b** to find Wally's viewing angle θ.

Problem solving without all the help

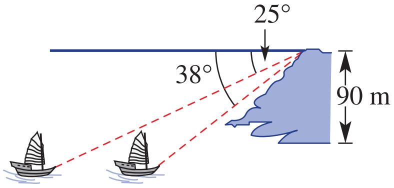
Solve these similar types of problems. You will need to draw detailed diagrams and split the problem into parts. Refer to the above two problems if you need help.

a An observer is 50 m horizontally from a hot air balloon. The angle of elevation to the top of the balloon is 60° and to the bottom of the balloon's basket is 40°. Find the total height of the balloon (to the nearest metre) from the base of the basket to the top of the balloon.

b A ship (at A) is 24 km due east of a lighthouse (L). The captain takes bearings from two landmarks, M and Q, which are due north and due south of the lighthouse respectively. The true bearings of M and Q from the ship are 322° and 244° respectively. How far apart are the two landmarks?



c From the top of a 90 m cliff the angles of depression of two boats in the water, both directly east of the light house, are 25° and 38° respectively. What is the distance between the two boats to the nearest metre?



d A person on a boat 200 m out to sea views a 40 m high castle wall on top of a 32 m high cliff. Find the viewing angle between the base and top of the castle wall from the person on the boat.

Design your own problem

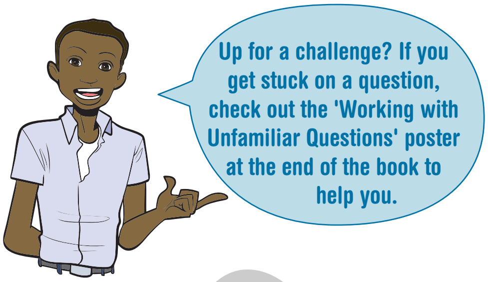
Design a problem similar to the ones above that involve a combination of triangles.

a Clearly write the problem.

b See if a friend can understand and solve your problem.

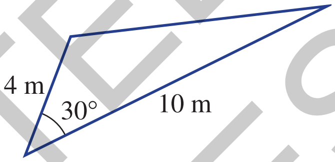
c Show a complete solution including all diagrams.

Problems and challenges



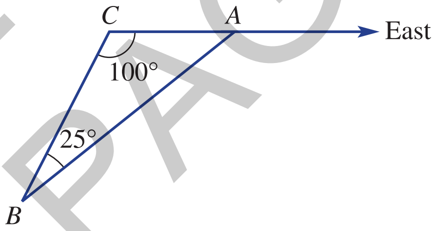
1 A right-angled isosceles triangle has area of 4 square units. Determine the exact perimeter of the triangle.

2 Find the area of this triangle using trigonometry.
Hint: insert a line showing the height of the triangle.



3 A rectangle *ABCD* has sides *AB* = *CD* = 34 cm. *E* is a point on *CD* such that *CE* = 9 cm and *ED* = 25 cm. *AE* is perpendicular to *EB*. What is the length of *BC*?

4 Find the bearing from *B* to *C* in this diagram.



5 Which is a better fit? A square peg in a round hole or a round peg in a square hole. Use area calculations and percentages to investigate.

6 Boat A is 20 km from port on a true bearing of 025° and boat B is 25 km from port on a true bearing of 070°. Boat B is in distress. What bearing (to the nearest degree) should boat A travel on to reach boat B?

7 For positive integers m and n such that $n < m$, the Pythagorean triples (like 3, 4, 5) can be generated using $a = m^2 - n^2$ and $b = 2mn$, where a and b are the two shorter sides of the right-angled triangle.

a Using the above formulas and Pythagoras' theorem to calculate the third side, generate the Pythagorean triples for:

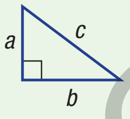
  i $m = 2, n = 1$ **ii** $m = 3, n = 2$

b Using the expressions for a and b and Pythagoras' theorem, find a rule for c (the hypotenuse) in terms of n and m.

Chapter summary

Pythagoras' theorem and trigonometry

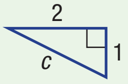
Pythagoras' theorem



$c^2 = a^2 + b^2$

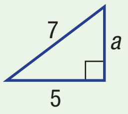
Finding the hypotenuse

$$\begin{aligned} c^2 &= 2^2 + 1^2 \\ &= 5 \\ c &= \sqrt{5} \\ &= 2.24 \text{ (to 2 decimal places)} \end{aligned}$$

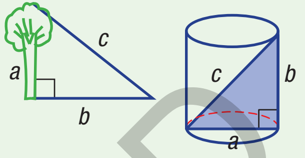


Shorter sides

$$\begin{aligned} a^2 + 5^2 &= 7^2 \\ a^2 &= 7^2 - 5^2 \\ &= 24 \\ \therefore\ a &= \sqrt{24} \end{aligned}$$



Applications

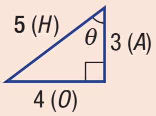


SOHCAHTOA

$\sin\theta = \dfrac{O}{H} = \dfrac{4}{5}$

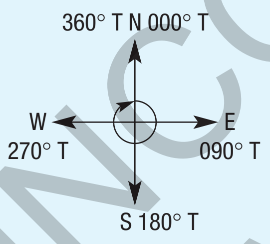
$\cos\theta = \dfrac{A}{H} = \dfrac{3}{5}$

$\tan\theta = \dfrac{O}{A} = \dfrac{4}{3}$



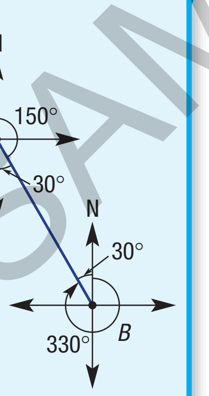
Bearings Ext

True bearings are measured clockwise from North.

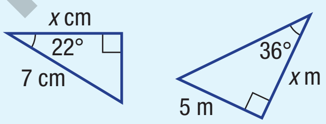


B is 150° T from *A*

A is 330° T from *B*



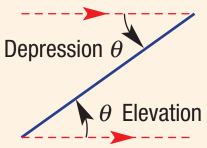
Finding side lengths



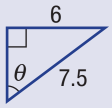
$$\begin{aligned} \cos 22° &= \frac{x}{7} \\ x &= 7\cos 22° \\ &= 6.49 \\ &\text{(to 2 d. p.)} \end{aligned}$$

$$\begin{aligned} \tan 36° &= \frac{5}{x} \\ x \times \tan 36° &= 5 \\ x &= \frac{5}{\tan 36°} \\ &= 6.88 \text{ (to 2 d. p.)} \end{aligned}$$

Elevation and depression



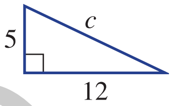
Finding angles



$$\begin{aligned} \sin\theta &= \frac{6}{7.5} \\ \theta &= \sin^{-1}\left(\frac{6}{7.5}\right) \\ &= 53.13° \text{ (to 2 d. p.)} \end{aligned}$$

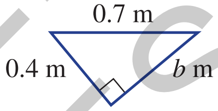
Multiple-choice questions

3A **1** For the right-angled triangle shown, the length of the hypotenuse can be found from the equation:



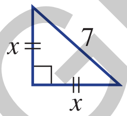
A $c^2 = 5^2 + 12^2$ **B** $c^2 = 5^2 - 12^2$
C $c^2 = 12^2 - 5^2$ **D** $c^2 = 5^2 \times 12^2$
E $(5 + 12)^2$

3B **2** For the right-angled triangle shown, the value of b is given by:



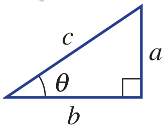
A $\sqrt{0.7^2 + 0.4^2}$ **B** $\sqrt{0.7^2 - 0.4^2}$
C $\sqrt{0.4^2 - 0.7^2}$ **D** $\sqrt{0.7^2 \times 0.4^2}$
E $\sqrt{(0.7 - 0.4^2)^2}$

3B **3** For the right-angled triangle shown:



A $x^2 = \frac{49}{2}$ **B** $7x^2 = 2$ **C** $x^2 = \frac{7}{2}$
D $x^2 + 7^2 = x^2$ **E** $x^2 = \frac{2}{7}$

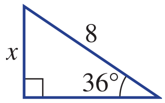
3E **4** For the triangle shown:



A $\sin\,\theta = \frac{a}{b}$ **B** $\sin\,\theta = \frac{c}{a}$ **C** $\sin\,\theta = \frac{a}{c}$
D $\sin\,\theta = \frac{b}{c}$ **E** $\sin\,\theta = \frac{c}{b}$

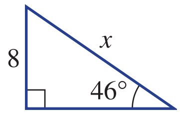
3F **5** The value of $\cos 46^\circ$ correct to four decimal places is:
A 0.7193 **B** 0.6947 **C** 0.594 **D** 0.6532 **E** 1.0355

3F **6** In the diagram the value of x, correct to two decimal places, is:



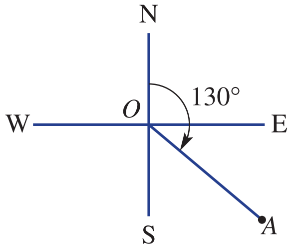
A 40 **B** 13.61 **C** 4.70
D 9.89 **E** 6.47

3G **7** The length of x in the triangle is given by:



A $8 \sin 46^\circ$ **B** $8 \cos 46^\circ$ **C** $\frac{8}{\cos 46^\circ}$
D $\frac{8}{\sin 46^\circ}$ **E** $\frac{\cos 46^\circ}{8}$

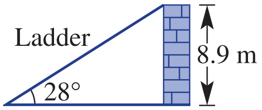
3J **8** The true bearing of A from O is 130°. The true bearing of O from A is:



A 050° **B** 220° **C** 310°
D 280° **E** 170°

3G

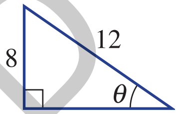
9 A ladder is inclined at an angle of 28° to the horizontal. If the ladder reaches 8.9 m up the wall, the length of the ladder correct to the nearest metre is:



A 19 m **B** 4 m **C** 2 m
D 10 m **E** 24 m

3H

10 The value of θ in the diagram, correct to two decimal places, is:



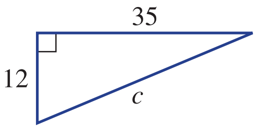
A 0.73° **B** 41.81° **C** 48.19°
D 33.69° **E** 4.181°

Short-answer questions

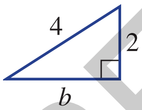
3A/B

1 Find the unknown length in these triangles. Give an exact answer.

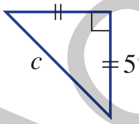
a



b

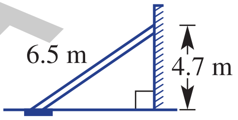


c



3B

2 A steel support beam of length 6.5 m is connected to a wall at a height of 4.7 m from the ground. Find the distance (to the nearest centimetre) between the base of the building and the point where the beam is joined to the ground.

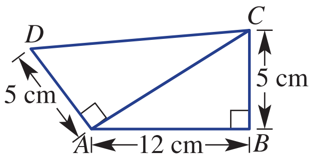


3A

3 For this double triangle, find:

a AC

b CD (correct to two decimal places).



3C

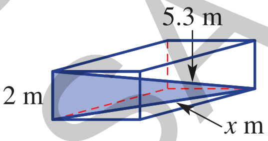
4 Two different cafés on opposite sides of an atrium in a shopping centre are respectively 10 m and 15 m above the ground floor. If the cafés are linked by a 20 m escalator, find the horizontal distance (to the nearest metre) across the atrium, between the two cafés.

3D

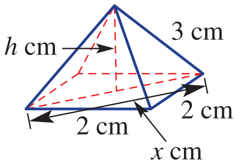
Ext

5 Find the values of the pronumerals in the three-dimensional objects shown below, correct to two decimal places.

a



b



3F

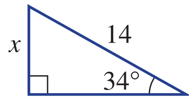
6 Find the value of each of the following, correct to two decimal places.

a sin 40° **b** tan 66° **c** cos 44°

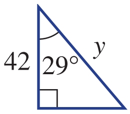
3F/G/H

7 Find the value of each pronumeral, correct to two decimal places.

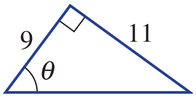
a



b

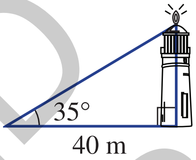


c



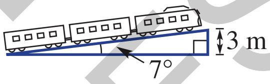
3I Ext

8 The angle of elevation of the top of a lighthouse from a point on the ground 40 m from its base is 35°. Find the height of the lighthouse to two decimal places.



3I Ext

9 A train travels up a slope, making an angle of 7° with the horizontal. When the train is at a height of 3 m above its starting point, find the distance it has travelled up the slope, to the nearest metre.

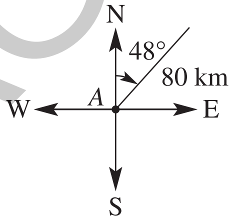


3J Ext

10 A yacht sails 80 km on a true bearing of 048°.

a How far east of its starting point is the yacht, correct to two decimal places?

b How far north of its starting point is the yacht, correct to two decimal places?



3I Ext

11 From a point on the ground, Geoff measures the angle of elevation of a 120 m tower to be 34°. How far from the base of the tower is Geoff, correct to two decimal places?

3J Ext

12 A ship leaves Coffs Harbour and sails 320 km east. It then changes direction and sails 240 km due north to its destination. What will the ship's true bearing be from Coffs Harbour when it reaches its destination, correct to two decimal places?

3I Ext

13 From the roof of a skyscraper, Aisha spots a car at an angle of depression of 51° from the roof of the skyscraper. If the skyscraper is 78 m high, how far away is the car from the base of the skyscraper, correct to one decimal place?

3F

14 Penny wants to measure the width of a river. She places two markers, *A* and *B*, 10 m apart along one bank. *C* is a point directly opposite marker *B*. Penny measures angle *BAC* to be 28°. Find the width of the river to one decimal place.

3I Ext

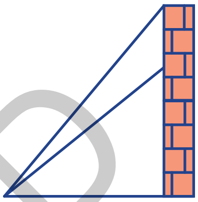
15 An aeroplane takes off and climbs at an angle of 15° to the horizontal, at 210 km/h along its flight path for 15 minutes. Find:

a the distance the aeroplane travels

b the height the aeroplane reaches, correct to two decimal places.

Extended-response questions

1 An extension ladder is initially placed so that it reaches 2 m up a wall. The foot of the ladder is 80 cm from the base of the wall.



a Find the length of the ladder, to the nearest centimetre, in its original position.

b Without moving the foot of the ladder, it is extended so that it reaches 1 m further up the wall. How far (to the nearest centimetre) has the ladder been extended?

c The ladder is placed so that its foot is now 20 cm closer to the base of the wall.

  i How far up the wall can the ladder length found in part **b** reach? Round to two decimal places.

  ii Is this further than the distance in part **a**?

Ext **2** From the top of a 100 m cliff, Skevi sees a boat out at sea at an angle of depression of 12°.

a Draw a diagram for this situation.

b Find how far out to sea the boat is to the nearest metre.

c A swimmer is 2 km away from the base of the cliff and in line with the boat. What is the angle of depression to the swimmer, to the nearest degree?

d How far away is the boat from the swimmer, to the nearest metre?

Ext **3** A pilot takes off from Amber Island and flies for 150 km at 040° T to Barter Island where she unloads her first cargo. She intends to fly to Dream Island but a bad thunderstorm between Barter and Dream islands forces her to fly off-course for 60 km to Crater Atoll on a bearing of 060° T. She then turns on a bearing of 140° T and flies for 100 km until she reaches Dream Island where she unloads her second cargo. She then takes off and flies 180 km on a bearing of 055° T to Emerald Island.

a How many extra kilometres did she fly trying to avoid the storm? Round to the nearest kilometre.

b From Emerald Island she flies directly back to Amber Island. How many kilometres did she travel on her return trip? Round to the nearest kilometre.